**Problem 1.** [Final Exam, Autumn 2007, 1] Consider the function
\[ f(x) := (3 + x)^{1/2}. \]
(a) Find the second Taylor polynomial \( T_2 \) of \( f \) based at \( b = 1 \).
(b) Use the Taylor polynomial you computed above to approximate \( \sqrt{3.7} \).
(c) Use Taylor’s inequality to find an upper bound for the error in your approximation above.

**Problem 2.** [Final Exam, Spring 2007, 1] Consider the function \( f(x) = x^3 + x \).
(a) Find the second Taylor polynomial \( T_2 \) of \( f \) based at \( b = 1 \).
(b) Use Taylor’s inequality to find an interval \( J \) around \( b \) such that the error \( |T_2(x) - f(x)| \) is less than 0.001 for all \( x \) in \( J \).

**Problem 3.** [Final Exam, Winter 2007, 1] Let \( f(x) = \frac{1}{5-x}, I = [-2, 2], \) and \( b = 0 \).
(a) Find the first Taylor polynomial for \( f(x) \) based on \( b \).
(b) Use Taylor’s inequality to give a bound for the error \( |f(x) - T_1(x)| \) on \( I \).
(c) Find an integer \( n \) such that the error \( |f(x) - T_n(x)| \) on \( I \) given by Taylor’s inequality is smaller than 0.05 and larger than 0.04.

**Problem 4.** [Final Exam, Winter 2008, 2] Use Taylor’s inequality to find \( n \) such that the Taylor polynomial of degree \( n \) centered at \( x = 0 \) for the function \( g(x) = e^{2x} \) approximates \( g(x) \) with accuracy 0.01 on the interval \( [-.5, 0] \).

**Problem 5.** [Final Exam, Spring 2008, 1] Consider the function
\[ f(x) = \sin(x - 4) + \cos(x - 4) + 4\sqrt{x}. \]
(a) Find the second Taylor polynomial \( T_2 \) of \( f(x) \) based at \( b = 4 \).
(b) Use the second Taylor polynomial \( T_2 \) to approximate \( f(4.1) \).
(c) Use Taylor’s inequality to find an upper bound for the error in your approximation above.
Solutions

Problem 1. (a) First, let us calculate the first two derivatives and the function itself at the point $x = b = 1$.

$$f(x) = (3 + x)^{1/2} \Rightarrow f(1) = 2;$$

$$f'(x) = \frac{1}{2} (3 + x)^{-1/2} \Rightarrow f'(1) = \frac{1}{4};$$

$$f''(x) = \frac{1}{2} \left( -\frac{1}{2} \right) (3 + x)^{-3/2} = -\frac{1}{4} (3 + x)^{-3/2} \Rightarrow f''(1) = -\frac{1}{32}.$$ 

Thus,

$$T_2(x) = f(1) + f'(1)(x - 1) + \frac{f''(1)}{2} (x - 1)^2 = 2 + \frac{1}{4}(x - 1) - \frac{1}{64}(x - 1)^2.$$ 

(b) $\sqrt{3.7} = \sqrt{3 + 0.7} = f(0.7) \approx T_2(0.7) = 2 - \frac{0.3}{4} - \frac{0.9}{64}$.

(c) The Taylor inequality gives us the following upper estimate:

$$\frac{M}{3!}|0.7 - 1|^3, \quad M := \max_{x \in [0.7, 1]} |f^{(3)}(x)|.$$ 

Let us find $M$:

$$f^{(3)}(x) = -\frac{1}{4} \left( -\frac{3}{2} \right) (3 + x)^{-5/2} = \frac{3}{8} (3 + x)^{-5/2}, \quad |f^{(3)}(x)| = \frac{3}{8} (3 + x)^{-5/2},$$

and this function decreases on $[0.7, 1]$, hence it attains its maximum at the left endpoint of this interval:

$$M = |f^{(3)}(0.7)| = \frac{3}{8} (3.7)^{-5/2}.$$ 

Thus, the upper bound for this error is

$$\frac{3}{8} (3.7)^{-5/2} \frac{(0.3)^3}{6} = \frac{(0.3)^3}{16 (3.7)^{5/2}}.$$ 

Note: You may leave the answers in this exact form. Do not waste your time on the final exam using a calculator to find the decimal approximation.

Problem 2. (a) Similarly, we find $f(1) = 2, f'(1) = (3x^2 + 1)|_{x=1} = 4, f''(1) = (6x)|_{x=1} = 6,$ and $T_2(x) = 2 + 4(x - 1) + 6(x - 1)^2/2 = 2 + 4(x - 1) + 3(x - 1)^2$.

(b) Let $J = [1 - \varepsilon, 1 + \varepsilon], \varepsilon > 0$ is unknown, the question of the problem is to find it. The error bound is

$$\frac{M}{3!}\varepsilon^3, \quad M := \max_{x \in [1 - \varepsilon, 1 + \varepsilon]} |f^{(3)}(x)|.$$ 

But $f^{(3)}(x) = 6$, this is a constant function, hence $M = 6$, and the error bound is simply $\varepsilon^3$. And $\varepsilon^3 < 0.001 = (0.1)^3 \Leftrightarrow \varepsilon < 0.1$, so you may take any $\varepsilon < 0.1$, e.g. 0.05 or 0.09.

Problem 3. (a) $f'(x) = 1/(5 - x)^2$, so $f(0) = 1/5, f''(0) = 1/25, T_1(x) = \frac{1}{5} + \frac{x^2}{25}$.

(b) $f''(x) = 2/(5 - x)^3, |f''(x)| = 2/(5 - x)^3$, this function increases on $I$, since the denominator $(5 - x)^3$ decreases on $I$; therefore, it attains its maximum on this interval at its right endpoint, 

$$M := \max_{x \in I} |f''(x)| = |f''(2)| = \frac{2}{27}.$$
The upper bound is \((M/2) \cdot 2^2 = 4/27\).

(c) It can be shown that for \(k = 1, 2, 3, \ldots\) we have
\[
f^{(k)}(x) = \frac{k!}{(5 - x)^{k+1}}.
\]
Therefore,
\[
M_k := \max_{x \in I} |f^{(k)}(x)| = \max_{x \in I} \frac{1}{(5 - x)^{k+1}} = \frac{1}{(5 - 2)^{k+1}} = \frac{k!}{3^{k+1}}.
\]
The upper bound for \(|f(x) - T_n(x)|\) is
\[
B_n := \frac{M_{n+1}}{(n+1)!} 2^{n+1} = \frac{(n+1)!}{3^{n+2}(n+1)!} 2^{n+1} = \frac{(2/3)^{n+1}}{3}.
\]
We must find all \(n\) such that \(0.04 < B_n < 0.05\). The sequence \(B_1, B_2, B_3, \ldots\) decreases. \(B_1 = \frac{4}{27} > 0.05, B_2 = \frac{8}{81} > 0.05, B_3 = \frac{16}{213} > 0.05, B_4 = \frac{32}{729} \in (0.04, 0.05), B_5 = \frac{64}{2187} < 0.04\). Thus, the only \(n\) such that \(0.04 < B_n < 0.05\) is \(n = 4\).

**Problem 4.** By the Taylor inequality, the upper bound is
\[
B_n := \frac{M_{n+1}}{(n+1)!} (0.5)^{n+1}, \quad M_k := \max_{x \in [-0.5,0]} |g^{(k)}(x)|.
\]
But \(g^{(k)}(x) = 2^k e^{2x}, |g^{(k)}(x)| = 2^k e^{2x}\), this function increases, hence this maximum is attained at the right endpoint: \(M_k = |g^{(k)}(0)| = 2^k\). Therefore,
\[
B_n = \frac{1}{(n+1)!}.
\]
The sequence \(B_1, B_2, \ldots\) decreases. \(B_3 = 1/24 > 0.01, B_4 = 1/120 < 0.01\). Hence we can take \(n = 4\).

Note: Of course, we can take a larger \(n\), but this is not advisable, since it would be harder to compute the Taylor approximation. But if you write e.g. \(n = 6\), this would be a correct answer.

**Problem 5.** (a) Let us find the values of this function and its first and second derivatives at \(b = 4\):
\[
f(x) = \sin(x - 4) + \cos(x - 4) + 4\sqrt{x} \Rightarrow f(4) = 9;
\]
\[
f'(x) = \cos(x - 4) - \sin(x - 4) + 2x^{-1/2} \Rightarrow f'(4) = 2;
\]
\[
f''(x) = -\sin(x - 4) - \cos(x - 4) - 3x^{-3/2} \Rightarrow f''(4) = -1 - \frac{1}{8} = -\frac{9}{8}.
\]
Therefore,
\[
T_2(x) = 9 + 2(x - 4) - \frac{9}{16}(x - 4)^2.
\]
(b) \(f(4.1) \approx T_2(4.1) = 9 + 0.2 - \frac{0.09}{16}\).
(c) The upper bound is \((M/3!)^3(4.1 - 4)^3, M := \max_{x \in [4,4.1]} |f^{(3)}(x)|\). For \(x \in [4,4.1]\),
\[
f^{(3)}(x) = -\cos(x - 4) + \sin(x - 4) + \frac{3}{2} x^{-5/2};
\]
\[
|f^{(3)}(x)| \leq |\cos(x - 4)| + |\sin(x - 4)| + \frac{3}{2} x^{-5/2} \leq 1 + 1 + \frac{3}{2} 4^{-5/2} = 2 + \frac{3}{64} < 3.
\]
Hence, \(M < 3\), and we may take the upper bound to be \((3/3!)(0.1)^3 = 0.0005\). (In fact, it is much smaller.)