Challenge Problems 3
Math 126E. 06/30/2011

Problem 1. [Midterm 1, Milakis, Winter 2009, 2] (a) Check if the planes
\[ x + 3y + z = 2 \] and \[ 2x + y - z = -1 \]
are parallel. If not, find a parametric equation of the line of intersection of the two planes.
(b) Find, correct to the nearest degree, the angle between these two planes.

Problem 2. [Midterm 1, Milakis, Winter 2009, 3] (a) Determine whether the lines
\[ \mathbf{r}_1(t) = \langle 2, -1, 0 \rangle + t \langle -1, 1, 1 \rangle \] and \[ \mathbf{r}_2(s) = \langle 1, 3, 0 \rangle + s \langle -2, -1, 3 \rangle \]
are parallel, skew or intersecting. If they intersect, find the point of intersection.
(b) Find (if exists) an equation of the plane that contains these lines.

Problem 3. [Midterm 1, Conroy, Spring 2007, 3] Find the equation of the plane containing
the line of intersection of the two planes
\[ x + y + z + 5 = 0 \] and \[ 3x + 2y - z + 2 = 0 \]
and the point \( (1, 2, 1) \).

Problem 4. [Midterm 1, Gunnarsson, Spring 2007, 5] Let \( \mathbf{v} = \langle 1, 3, -1 \rangle \) and \( \mathbf{r}_0 = \langle 1, 1, 1 \rangle \)
and consider the line given by \( \mathbf{r} = \mathbf{r}_0 + t\mathbf{v} \), in vector form. Also, consider the plane given by
\[ x + 2y + 2z + 2 = 0. \]
(a) Show that the line and the plane are not parallel.
(b) Find the point on the line at distance 3 from the plane.

Problem 5. [Final Exam, Spring 2010, 4] (a) Find the equation of the plane containing the line
\[ x = 1 + 3t, \quad y = 2 + 2t, \quad z = 3 + t \]
and the point \( (0, 2, 5) \).
(b) Write the equation of the line of intersection of the two planes defined by \( 2x - z = 0 \) and \( x + y + z = 1 \).
Solutions

Problem 1. (a) These planes are not parallel, since their normal vectors, $\mathbf{n}_1 := <1, 3, 1>$ and $\mathbf{n}_2 := <2, 1, -1>$, are not parallel. One can verify this using their cross product: $<1, 3, 1> \times <2, 1, -1> = <3 \cdot (-1) - 1 \cdot 1, 1 \cdot 2 - (-1) \cdot 1, 1 \cdot 1 - 3 \cdot 2> = <-4, 3, -5> \neq \mathbf{0}$. Recall that any two nonzero vectors are parallel if and only if their cross product equals $\mathbf{0}$.

To obtain the parametric equations of the line of intersection, we need: (1) to find a directional vector of this line and (2) to find a certain point on this line.

(1) This line lies on the first plane; therefore, it is orthogonal to the normal vector $\mathbf{n}_1$. This line lies on the second plane; therefore, it is orthogonal to the normal vector $\mathbf{n}_2$. Their cross product $\mathbf{n}_1 \times \mathbf{n}_2$ is also orthogonal to $\mathbf{n}_1$ and $\mathbf{n}_2$; therefore, we can take this vector as $\mathbf{v}$, the directional vector of this line. We have already calculated this cross product, it is $<-4, 3, -5>$.

(2) We have a system of equations:

$$
x + 3y + z = 2, \quad 2x + y - z = -1.
$$

This system contains three variables and two equations. Such systems generally have infinitely many solutions. This one is not an exception. But we need to find only one point (doesn’t matter which one). So set $z = 0$, then we have:

$$
x + 3y = 2, \quad 2x + y = -1.
$$

Therefore, $x = 2 - 3y$, $2(2 - 3y) + y = -1$, $4 - 5y = -1$, $5y = 5$, $y = 1$, $x = -1$. The point $(-1, 1, 0)$ lies on this line. And we immediately obtain the parametric equations: $x = -1 - 4t, y = 1 + 3t, z = -5t$.

(b) The angle between these planes is the angle $\theta$ between the normal vectors, if $\theta \leq \pi/2$ (or $\pi - \theta$, if $\theta > \pi/2$). But $\mathbf{n}_1 \cdot \mathbf{n}_2 = 1 \cdot 2 + 3 \cdot 1 + 1 \cdot (-1) = 4$, $|\mathbf{n}_1| = \sqrt{1^2 + 3^2 + 1^2} = \sqrt{11}$, $|\mathbf{n}_2| = \sqrt{2^2 + 1^2 + (-1)^2} = \sqrt{6}$. Thus

$$
\cos \theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|} = \frac{4}{\sqrt{11}\sqrt{6}} = \frac{4}{\sqrt{66}}.
$$

Since $\cos \theta > 0$, $\theta < \pi/2$ and the angle between the planes is $\theta$. The approximate value of $\theta$ is $61^\circ$.

Problem 2. (a) First of all, these lines are not parallel, since their directional vectors, $\mathbf{v}_1 := <-1, 1, 1>$ and $\mathbf{v}_2 := <-2, -1, 3>$, are not parallel. Similarly to Problem 1, this rather obvious fact may be proved by taking the cross product of these vectors:

$$
\mathbf{v}_1 \times \mathbf{v}_2 = <1 \cdot 3 - 1 \cdot (-1), 1 \cdot (-2) - (-1) \cdot 3, (-1) \cdot (-1) - (-2) \cdot 1> = <4, 1, 3>.
$$

and observing that this cross product is not equal to the zero vector $\mathbf{0}$.

Let us rewrite the equations of these lines in parametric form. First line:

$$
x = 2 - t, \quad y = -1 + t, \quad z = t.
$$

Second line:

$$
x = 1 - 2s, \quad y = 3 - s, \quad z = 3s.
$$

To find whether they have any point of intersection, we need to solve the following system of equations:

$$
2 - t = 1 - 2s, \quad -1 + t = 3 - s, \quad t = 3s.
$$
This system contains three equations and two variables. The number of equations exceeds the number of variables, and in general such systems do not have any solution. But this system does have a solution: plugging in $3s$ instead of $t$ in the first two equations, we immediately obtain:

$$2 - 3s = 1 - 2s, \quad -1 + 3s = 3 - s.$$  

It is easy to check that both of these equations have the same root $s = 1$. If they had different roots, the system above would not have any solutions and the lines would not intersect. But these lines do intersect, and the point of intersection is $r_2(1) = <-1, 2, 3>.

(b) Since these lines intersect, there exists a plane that contains these lines. (Two lines are contained in a certain plane if and only if they intersect or are parallel.) This plane contains the point $(-1, 2, 3)$ and is parallel to the vectors $v_1$ and $v_2$. Hence the following vector is a normal vector to the plane: $v_1 \times v_2 = <-4, 1, 3>$. We can immediately write the equation of this plane:

$$4(x + 1) + (y - 2) + 3(z - 3) = 0.

**Problem 3.** First, let us find the line of intersection of these two planes. We need to solve this system of equations:

$$x + y + z + 5 = 0, \quad 3x + 2y - z + 2 = 0.$$  

Similarly to Problem 1, let us eliminate one of the variables, e.g. $z$, denoting it as a parameter: $z = t$. We obtain:

$$x + y = -5 - t, \quad 3x + 2y = -2 + t.$$  

Hence

$$x = 3x + 2y - 2(x + y) = -2 + t - 2(-5 - t) = 8 + 3t,$$

$$y = -5 - t - y = -5 - t - (8 + 3t) = -13 - 4t.$$  

Thus, parametric equations of the line of intersection:

$$x = 8 + 3t, \quad y = -13 - 4t, \quad z = t.$$  

The plane that contains this line and the point $P(1, 2, 1)$ is parallel to the directional vector $<3, -4, 1>$ of this line. It is also parallel to $P_0P$, where $P_0$ is any point on this line. We can set, e.g., $t = 0$ (we might as well choose another $t$), then $P_0 = <-8, -13, 0>$, $P_0P = <-7, 15, 1>$.  

So this line is parallel to $<3, -4, 1>$ and $<-7, 15, 1>$. Hence their cross product is a normal vector: $<3, -4, 1> \times <-7, 15, 1> = <-15 \cdot 1 - 1 \cdot (-4), 1 \cdot 3 - (-7) \cdot 1, (-7) \cdot 4 - 3 \cdot 15> = <-19, 10, -17>.$ This plane contains the point $P(1, 2, 1)$ and its normal vector is $<19, 10, -17>$. Thus, its equation is

$$19(x - 1) + 10(y - 2) - 17(z - 1) = 0.$$

**Problem 4.** (a) A line and a plane are parallel if and only if the directional vector of the line is orthogonal to the normal vector of the plane. But $<1, 3, -1>$ is the directional vector of this line, and $<1, 2, 2>$ is the normal vector of this plane. Their dot product is $<1, 3, -1> \cdot <1, 2, 2> = 1 \cdot 1 + 3 \cdot 2 + (-1) \cdot 2 = 5 \neq 0$, hence they are not orthogonal; and the given line and plane are not parallel.

(b) The distance from the point $(x_0, y_0, z_0)$ to the plane $x + 2y + 2z + 2 = 0$ is equal to

$$d = \frac{|x_0 + 2y_0 + 2z_0 + 2|}{\sqrt{1^2 + 2^2 + 2^2}} = \frac{|x_0 + 2y_0 + 2z_0 + 2|}{3}.$$  

$$3$$
The parametric equations of the line:
\[ x = 1 + t, \quad y = 1 + 3t, \quad z = 1 - t. \]

Hence the distance from the point \((1 + t, 1 + 3t, 1 - t)\) to the plane is
\[
\frac{1}{3} |(1 + t) + 2(1 + 3t) + 2(1 - t) + 2| = \frac{1}{3} |7 + 5t|.
\]

It is equal to 3 if and only if
\[ |7 + 5t| = 9, \quad 7 + 5t = \pm 9, \quad 5t = 2 \text{ or } -16, \quad t = \frac{2}{5} \text{ or } -\frac{16}{5}. \]

Thus, there are two points on the line at distance 3 from this plane:
\[
\left(\frac{7}{5}, \frac{11}{5}, \frac{3}{5}\right), \quad \left(-\frac{11}{5}, -\frac{43}{5}, \frac{21}{5}\right).
\]

**Problem 5.** (a) It suffices to find a normal vector to the plane; then we can immediately write the equation, because we already have a point \(P = (0, 2, 5)\) on this plane. A normal vector can be obtained as a cross product of two vectors on this plane. The first vector is \(v_1 = <3, 2, 1>\) (the directional vector of this line; since the line lies on this plane, this directional vector also lies on this plane). The second vector is \(v_2 = PQ\), where \(Q\) is any point on this line. Take, e.g. \(t = 0\) and get \(Q = (1, 2, 3)\). Then \(v_2 = PQ = <1, 0, -2>\).

And the normal vector is \(n = v_1 \times v_2 = <-4, 7, -2>\). So the equation of this plane is
\[ -4x + 7(y - 2) - 2(z - 5) = 0. \]

(b) The line of intersection lies on the first plane; therefore, it is orthogonal to the normal vector \(n_1 = <2, 0, -1>\) of this plane. The line of intersection lies on the second plane; therefore, it is orthogonal to the normal vector \(n_2 = <1, 1, 1>\) of this plane. The vector \(v = n_1 \times n_2 = <1, -3, 2>\) is also orthogonal to both normal vectors; therefore, it is a directional vector of this line.

It suffices to find some point on this line. Set \(z = 0\) and solve the system of equations
\[ 2x = 0, \quad x + y = 1. \]
We get: \(x = 0, \quad y = 1\). So \((0, 1, 0)\) lies on both planes, therefore, on this line. Thus, the equation of this line is
\[ x = t, \quad y = 1 - 3t, \quad z = 2t. \]