Problem 1. [Midterm 2, Milakis, Winter 2009, 5] Let

\[ D := \{(x, y) \mid 1 \leq x \leq 2, \ln x \leq y \leq e^x \}. \]

Compute the area of \( D \).

Problem 2. [Midterm 2, Perkins, Winter 2009, 3a] Evaluate the following integral:

\[ \int \int_{R} xy \sin(x^2y) \, dxdy, \quad R := [0, 1] \times [0, \pi/2]; \]

Problem 3. [Final Exam, Winter 2007, 3] (a) Draw a picture of the region \( R \) bounded by the circles \( x^2 + y^2 = 25 \) and \( x^2 + y^2 = 16 \) in the first quadrant. Label at least two points.

(b) Evaluate the double integral

\[ \int \int_{R} x + \sqrt{x^2 + y^2} \, dxdy. \]

Problem 4. [Final Exam, Spring 2010, 6] Find the volume of the solid that lies under the plane \( 3x + 2y + z = 12 \) and above the rectangle \( R = [0, 1] \times [2, 3] \).

Problem 5. [Final Exam, Spring 2008, 7] Evaluate the integral

\[ \int_{0}^{2} \int_{0}^{4-x^2} \frac{xe^{2y}}{4-y} \, dydx. \]

Problem 6. [Final Exam, Autumn 2007, 10] Consider the region \( R \) bounded by a semi-circle of radius 2, a semi-circle of radius 1, and the x-axis (this \( R \) lies in the region \( \{ y > 0 \} \)). Compute the average value of the function

\[ f(x, y) = e^{-x^2-y^2} \]

over the region \( R \).
Solutions

Problem 1. The area of $D$ is

$$\iint_{D} 1 \, dx \, dy.$$ 

By Fubini’s Theorem, this integral equals

$$\int_{1}^{2} \left[ \int_{\ln x}^{e^x} 1 \, dy \right] \, dx = \int_{1}^{2} (e^x - \ln x) \, dx.$$ 

But

$$\int_{1}^{2} e^x \, dx = e^x \bigg|_{x=1}^{x=2} = e^2 - e.$$ 

How to compute the integral of $\ln x$? We need to find the antiderivative of this function. One common mistake was that

$$\int_{1}^{2} \ln x \, dx = \frac{1}{x} \bigg|_{x=2}^{x=1}.$$ 

But this is not true, since $(1/x)' \neq \ln x$! To find the antiderivative of $\ln x$, we need to integrate by parts:

$$\int \ln x \, dx = \int x' \ln x \, dx = x \ln x - \int x (\ln x)' \, dx = x \ln x - \int \frac{1}{x} \, dx = x \ln x - x.$$ 

Hence $(x \ln x - x)' = \ln x$, and by the Fundamental Theorem of Calculus

$$\int_{1}^{2} \ln x \, dx = (x \ln x - x) \bigg|_{x=1}^{x=2} = (2 \ln 2 - 2) - (\ln 1 - 1) = 2 \ln 2 - 1 = \ln 4 - 1.$$ 

Hence the area of $D$ is

$$e^2 - e - 2 \ln 2 - 1.$$ 

Problem 2. First, find this one-variable integral:

$$\int_{0}^{1} xy \sin(x^2 y) \, dx = \int_{0}^{y} \sin \frac{t \, dt}{2},$$ 

where we have changed the variables: $t = x^2 y$, $0 \leq t \leq y$, $dt = 2xy \, dx$. And

$$\int_{0}^{y} \sin \frac{t \, dt}{2} = \frac{1}{2} (-\cos t) \bigg|_{t=0}^{t=y} = \frac{1}{2} (1 - \cos y).$$ 

Hence by Fubini’s Theorem

$$\iint_{R} xy \sin(x^2 y) \, dxdy = \int_{0}^{\pi/2} \left[ \int_{0}^{1} xy \sin(x^2 y) \, dx \right] \, dy = \int_{0}^{\pi/2} \frac{1}{2} (1 - \cos y) \, dy =$$
Problem 3. (b) In polar coordinates, \( D = \{ 4 \leq r \leq 5, 0 \leq \theta \leq \pi/2 \} \). Also, \( \sqrt{x^2 + y^2} = r, \ x = r \cos \theta \). Let us rewrite this integral in polar coordinates:

\[
\int_0^{\pi/2} \int_0^5 (r \cos \theta + r) r dr d\theta - \int_0^{\pi/2} \int_0^4 (1 + \cos \theta) d\theta \int_0^5 r^2 dr = (\theta + \sin \theta) \bigg|_0^{\pi/2} \frac{r^3}{3} \bigg|_{r=4}^{r=5} = \left( \frac{\pi}{2} + 1 \right) \left( \frac{125}{3} - \frac{64}{3} \right) = \frac{61}{6} \pi + \frac{61}{3}.
\]

Problem 4. This body is between \( z = 0 \) and \( z = 12 - 3x - 2y \) and above the rectangle \( R \). So its volume is

\[
\int \int_R (12 - 3x - 2y) \, dA = \int_2^3 \int_0^1 (12 - 3x - 2y) \, dx \, dy.
\]

But

\[
\int_0^1 (12 - 3x - 2y) \, dx = \left( 12x - \frac{3x^2}{2} - 2xy \right) \bigg|_{x=0}^{x=1} = 12 - \frac{3}{2} - 2y = \frac{21}{2} - 2y.
\]

Therefore,

\[
\int_2^3 \int_0^1 (12 - 3x - 2y) \, dx \, dy = \int_2^3 \left( \frac{21}{2} - 2y \right) \, dy = \left( \frac{21}{2} y - \frac{1}{2} y^2 \right) \bigg|_{y=2}^{y=3} = \frac{21}{2} - 5 = \frac{11}{2}.
\]

Problem 5. Let us change the order of integration. \( y \leq 4 - x^2 \leftrightarrow x^2 \leq 4 - y \leftrightarrow x \leq \sqrt{4 - y} \). Therefore, the domain of integration is \( 0 \leq y \leq 4, \ 0 \leq x \leq \sqrt{4 - y} \). And the integral is equal to

\[
\int_0^4 \left( \int_0^{\sqrt{4-y}} \frac{e^{2y}}{4-y} \, dx \right) \, dy = \int_0^4 \frac{x^2}{2} \bigg|_{x=0}^{x=\sqrt{4-y}} \frac{e^{2y}}{4-y} \, dy = \int_0^4 \frac{4-y}{2} \frac{e^{2y}}{4-y} \, dy = \frac{1}{2} \int_0^4 e^{2y} \, dy = \frac{1}{2} \left( e^{4y} \bigg|_{y=0}^{y=4} - \frac{1}{4} \right) = e^8 - \frac{1}{4}.
\]

Problem 6. The area \( A \) of this region \( R = \{ 0 \leq \theta \leq \pi, \ 1 \leq r \leq 2 \} \) is the difference of the area of two semi-discs, one with radius 2, the other with radius 1. The area of a semi-disc with radius 1 is \( \frac{1}{2} \pi/2 = \pi/2 \); the area of a semi-disc with radius 2 is \( \frac{1}{2} \pi/2 = 2\pi \). The difference between them is \( 3\pi/2 \). So \( A = 3\pi/2 \).

The integral \( I \) of this function \( e^{-x^2-y^2} = e^{-r^2} \) over \( R \) can be rewritten as

\[
\int_0^\pi \int_0^2 e^{-r^2} r dr d\theta.
\]
But (after changing variables \( u = r^2, \ du = 2rd\theta \))

\[
\int_{1}^{2} e^{-r^2} r \, dr = \int_{1}^{4} e^{-u} \frac{du}{2} = \frac{1}{2} \left. (-e^{-u}) \right|_{u=2}^{u=1} = \frac{1}{2} (e^{-1} - e^{-4}).
\]

Therefore,

\[
I = \int_{0}^{\pi} \int_{1}^{2} e^{-r^2} r \, dr \, d\theta = \pi \frac{1}{2} (e^{-1} - e^{-4}) = \frac{\pi}{2} (e^{-1} - e^{-4}).
\]

And the average value is \( I/A = \frac{1}{3} (e^{-1} - e^{-4}). \)