1. \( \mathbf{v} \times \mathbf{v} = 0 \) only if \( \mathbf{v} = 0 \).
2. \( \mathbf{v} \cdot \mathbf{v} = 0 \) only if \( \mathbf{v} = 0 \).
3. \( \mathbf{u} \times (\mathbf{v} \cdot \mathbf{w}) \) makes sense.
4. \( \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) \) makes sense.
5. \( \mathbf{v} \cdot \mathbf{w} \) is a vector that is perpendicular to both \( \mathbf{v} \) and \( \mathbf{w} \).
6. \( \mathbf{v} \times \mathbf{w} \) is a vector that is perpendicular to both \( \mathbf{v} \) and \( \mathbf{w} \).
7. For all vectors \( \mathbf{v} \) and \( \mathbf{w} \) we have \( (\mathbf{v} - \mathbf{w}) \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{v} \cdot \mathbf{v} - \mathbf{w} \cdot \mathbf{w} \).
8. For all vectors \( \mathbf{v} \) and \( \mathbf{w} \) we have \( (\mathbf{v} - \mathbf{w}) \times (\mathbf{v} + \mathbf{w}) = \mathbf{v} \times \mathbf{v} - \mathbf{w} \times \mathbf{w} \).

Problem 2. [Midterm 1, Pevtsova, Winter 2007, 4] Check whether the points \((1, 2, 3), (-2, 5, 7), \) and \((-5, 8, 11)\) lie on the same line.

Problem 3. [Midterm 1, Pevtsova, Winter 2007, 5] Find the angle between the two main diagonals of a unit cube.
(A unit cube is a cube with the vertices \((0, 0, 0), (1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 0), (1, 0, 1), (0, 1, 1), (1, 1, 1), \) the main diagonals are the diagonals connecting the vertex \((0, 0, 0)\) with the vertex \((1, 1, 1)\), and the vertex \((1, 0, 0)\) with the vertex \((0, 1, 1)\)).

Problem 4. [Midterm 1, Conroy, Autumn 2010, 1] Find the angle between the vectors \( <3, 4, -1> \) and \( <5, 2, 8> \).

Problem 5. [Midterm 1, Gunnarsson, Spring 2007, 2] Consider the two vectors \( \mathbf{a} = <1, 2, 3> \) and \( \mathbf{b} = <2, 3, 4> \). Calculate the following:
(a) The cosine of the angle between \( \mathbf{a} \) and \( \mathbf{b} \);
(b) \( \mathbf{a} \times \mathbf{b} \);
(c) The area of the parallelogram with corner points \( P(0, 0, 0), Q(1, 2, 3), R(2, 3, 4), S(3, 5, 7) \).


Solutions

Problem 1. 1. FALSE. For ANY vector \( \mathbf{v} \), we have: \( \mathbf{v} \times \mathbf{v} = \mathbf{0} \).
2. TRUE. If \( \mathbf{v} \cdot \mathbf{v} = 0 \), then \( |\mathbf{v}|^2 = 0 \), \( |\mathbf{v}| = 0 \), and the only vector with magnitude 0 is the zero vector.
3. FALSE. \( \mathbf{v} \cdot \mathbf{w} \) is a scalar, and you cannot cross-multiply a vector by a scalar.
4. TRUE.
5. FALSE. \( \mathbf{v} \cdot \mathbf{w} \) is a scalar.
6. TRUE.
7. TRUE. \( (\mathbf{v} - \mathbf{w}) \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{v} \cdot \mathbf{v} - \mathbf{w} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w} - \mathbf{w} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{v} - \mathbf{w} \cdot \mathbf{w} \), because \( \mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v} \).
8. FALSE. \( (\mathbf{v} - \mathbf{w}) \times (\mathbf{v} + \mathbf{w}) = \mathbf{v} \times \mathbf{v} - \mathbf{w} \times \mathbf{v} + \mathbf{v} \times \mathbf{w} - \mathbf{w} \times \mathbf{w} = \mathbf{v} \times \mathbf{v} - \mathbf{w} \times \mathbf{w} + 2 \mathbf{v} \times \mathbf{w} \neq \mathbf{v} \times \mathbf{v} - \mathbf{w} \times \mathbf{w} \), because \( \mathbf{v} \times \mathbf{w} \) is not necessarily \( \mathbf{0} \).

Problem 2. Let \( P = (1, 2, 3) \), \( Q = (-2, 5, 7) \), and \( R = (-5, 8, 11) \). These points lie on the same line if and only if the vectors \( \mathbf{a} = \overrightarrow{PQ} = < -3, 3, 4 > \) and \( \mathbf{b} = \overrightarrow{QR} = < -3, 3, 4 > \) are parallel. But, obviously, they are equal, so they are parallel. (You might as well take the cross product in case of different numbers, when it is not so obvious.)

Problem 3. It is fairly obvious that the diagonals intersect at the center of the cube: \( M = (1/2, 1/2, 1/2) \). Of course, one can verify this by the following direct computation (but you do not need to write this at an exam).

The equations of the line containing the first diagonal are \( x = t, y = t, z = t \) (since it passes through the point \( (0, 0, 0) \) and its directional vector is \( < 1-0, 1-0, 1-0 > = < 1, 1, 1 > \)). Similarly, the equations of the line containing the second diagonal are \( x = 1-s, y = s, z = s \). They intersect at the point where \( t = s \) and simultaneously \( t = 1-s \), i.e. \( s = 1-s, s = 1/2 \); this point is \( (1-1/2 = 1/2, 1/2, 1/2) \).

Let \( O = (0, 0, 0) \), \( A = (1, 0, 0) \). The angle between these diagonals is the angle \( \theta = \angle OMA \), if \( \theta \leq \pi/2 \), or \( \pi - \theta \), if \( \theta > \pi/2 \). Therefore, we need to find \( \theta \). But

\[
\cos \theta = \frac{\mathbf{MO} \cdot \mathbf{MA}}{|\mathbf{MO}| |\mathbf{MA}|}.
\]

Since \( \mathbf{MA} = < 1/2, -1/2, -1/2 > \), \( \mathbf{MO} = < -1/2, -1/2, -1/2 > \), we have: \( \mathbf{MO} \cdot \mathbf{MA} = 1/4 \), \( |\mathbf{MO}| = |\mathbf{MA}| = \sqrt{3}/2 \). Thus

\[
\cos \theta = \frac{1/4}{(\sqrt{3}/2) \cdot (\sqrt{3}/2)} = \frac{1/4}{3/4} = \frac{1}{3}.
\]

Since \( \cos \theta > 0 \), \( \theta < \pi/2 \) and the angle between the two diagonals is \( \cos^{-1}(1/3) \approx 1.23 \approx 70.5^\circ \).

Problem 4. Let \( \mathbf{v} = < 3, 4, -1 > \) and \( \mathbf{w} = < 5, 2, 8 > \). Then \( \mathbf{v} \cdot \mathbf{w} = 3\cdot5 + 4\cdot2 + (-1)\cdot8 = 15 \), and the magnitudes of these vectors are: \( |\mathbf{v}| = \sqrt{3^2 + 4^2 + (-1)^2} = \sqrt{26} \), \( |\mathbf{w}| = \sqrt{5^2 + 2^2 + 8^2} = \sqrt{93} \). So if \( \theta \) is the angle between these vectors, we have:

\[
\cos \theta = \frac{\mathbf{v} \cdot \mathbf{w}}{|\mathbf{v}||\mathbf{w}|} = \frac{15}{\sqrt{26}\sqrt{93}}, \quad \theta = \arccos \left( \frac{15}{\sqrt{26}\sqrt{93}} \right).
\]

Problem 5. (a) \( \mathbf{a} \cdot \mathbf{b} = 1\cdot2 + 2\cdot3 + 3\cdot4 = 20 \), \( |\mathbf{a}| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14} \), \( |\mathbf{b}| = \sqrt{2^2 + 3^2 + 4^2} = \sqrt{29} \). Thus, if \( \theta \) is the angle between \( \mathbf{a} \) and \( \mathbf{b} \), we have:

\[
\cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \frac{20}{\sqrt{14}\sqrt{29}}.
\]

(b) \( \mathbf{a} \times \mathbf{b} = < 2\cdot4 - 3\cdot3, 3\cdot2 - 4\cdot1, 1\cdot3 - 2\cdot2 > = < -1, 2, -1 > \).

(c) This area is \( |\mathbf{PQ} \times \mathbf{PR}| = |\mathbf{a} \times \mathbf{b}| = \sqrt{(-1)^2 + 2^2 + (-1)^2} = \sqrt{6} \).