Problem 1. [Final Exam, Spring 2008, 5b] A curve in the $xy$-plane, called a cardioid, is determined by the polar equation
\[ r = 1 + \cos \theta. \]
Find the area of the region bounded by the $x$-axis and the cardioid $r = 1 + \cos \theta$ from $\theta = 0$ to $\theta = \pi$.

Problem 2. [Final Exam, Winter 2009, 8] Let
\[ D = \{(x, y) \in \mathbb{R}^2 \mid 4 \leq x^2 + y^2 \leq 4x, \; y \geq 0\}. \]
(a) Draw a careful picture for the domain $D$.
(b) Compute the area of $D$.

Problem 3. [Final Exam, Spring 2008, 8] A lamina occupies the region in the $xy$-plane bounded by the lines $x = 1$, $x = 2$, $y = ax$, and $y = 2ax$ for some positive number $a$. The lamina has density function $\rho(x, y) = \frac{1}{x} + \frac{1}{y}$. Find the value of $a$ that minimizes the mass of the lamina.

Problem 4. [Midterm 2, Taggart, Spring 2011, 5] The boundary of a lamina consists of the semicircles
\[ y = \sqrt{1 - x^2} \quad \text{and} \quad y = \sqrt{25 - x^2} \]
and the portions of the $x$-axis that join them.
The density of the lamina at any point is inversely proportional to its distance from the origin. That is, there is a constant $k$ such that, the density of the lamina at the point $(x, y)$ is
\[ \rho(x, y) = \frac{k}{\sqrt{x^2 + y^2}}. \]
Find the center of mass of the lamina.
(You may use the fact that, by symmetry, the center of mass is on the $y$-axis.)

Problem 5. [Final Exam, Spring 2007, 10] (a) Draw the picture of the region $R$ between the curves $r = 2 \cos \theta$ and $r = 2(1 + \cos \theta)$.
(b) Evaluate the area of $R$:
\[ A(R) = \iint_R 1 dA. \]

Problem 6. [Midterm 2, Perkins, Winter 2009, 3b] Evaluate the following integral:
\[ \iint_D y^2 e^{xy} dxdy, \; D := \{(x, y) \mid 0 \leq y \leq 3, \; 0 \leq x \leq y\}. \]
Solutions

Problem 1. If \( D = \{0 \leq r \leq 1 + \cos \theta, \ 0 \leq \theta \leq \pi\} \) is this region, then its area is

\[
\iint_D 1 \, dx \, dy = \int_0^\pi \int_0^{1+\cos \theta} r \, dr \, d\theta = \int_0^\pi \frac{1}{2} (1 + \cos \theta)^2 \, d\theta.
\]

But

\[
\frac{1}{2} (1 + \cos \theta)^2 = \frac{1}{2} (1 + 2 \cos \theta + \cos^2 \theta) = \frac{1}{2} (1 + 2 \cos \theta + \frac{1}{2} (1 + \cos 2\theta)) = \frac{3}{4} + \cos \theta + \frac{1}{4} \cos 2\theta.
\]

Since

\[
\int_0^\pi \cos \theta \, d\theta = \sin \theta \bigg|_{\theta=0}^{\theta=\pi} = 0, \quad \int_0^\pi \cos 2\theta \, d\theta = \frac{1}{2} \sin 2\theta \bigg|_{\theta=0}^{\theta=\pi} = 0,
\]

we have:

\[
\iint_D 1 \, dx \, dy = \int_0^\pi \frac{3}{4} \, d\theta = \frac{3}{4} \pi.
\]

Problem 2. (b) In polar coordinates, \( x^2 + y^2 = r^2, \ x = r \cos \theta, \) and

\[
D = \{4 \leq r^2 \leq 4 \cos \theta, \ 0 \leq \theta \leq \pi\} = \{2 \leq r \leq 4 \cos \theta, \ 0 \leq \theta \leq \pi\} = \{2 \leq r \leq 4 \cos \theta, \ \cos \theta \geq 1/2, \ 0 \leq \theta \leq \pi/3\}.
\]

The area of this domain is

\[
\iint_D 1 \, dx \, dy = \int_0^{\pi/3} \int_2^4 \cos \theta \, r \, dr \, d\theta = \int_0^{\pi/3} \frac{1}{2} (16 \cos^2 \theta - 4) \, d\theta.
\]

But \((16 \cos^2 \theta - 4)/2 = 8 \cos^2 \theta - 2 = 4(1 + \cos 2\theta) - 2 = 2 + 4 \cos 2\theta\). Therefore, this integral is equal to

\[
(2 \sin 2\theta + 2\theta)_{\theta=\pi/3}^{\theta=0} = 2 \sin (2\pi/3) - 2 \sin 0 + \frac{2\pi}{3} = \sqrt{3} + \frac{2\pi}{3}.
\]

Problem 3. First, let us calculate the mass \( m(a) \) of the lamina for every fixed value of the parameter \( a \).

\[
m(a) = \int_1^2 \int_{ax}^{2ax} \rho(x,y) \, dy \, dx = \int_1^2 \int_{ax}^{2ax} \left( \frac{1}{x} + \frac{1}{y^2} \right) \, dy \, dx = \int_1^2 \left( \frac{y}{x} - \frac{1}{y} \right)_{y=ax}^{2ax} \, dx = \int_1^2 \left( a + \frac{1}{2ax} \right) \, dx = \left( ax + \frac{1}{2a} \log x \right)_{x=1}^{x=2} = a + \frac{\log 2}{2a}.
\]
When does this function attain its minimum? Use differential calculus of one variable:

\[ m'(a) = 1 - \frac{\log 2}{2} \frac{1}{a^2} = 0 \Rightarrow a^2 = \frac{\log 2}{2} \Rightarrow a = \sqrt{\frac{\log 2}{2}}. \]

**Problem 4.** Suppose \( m \) is the center of mass of the lamina, \((x_c, y_c)\) is its center of mass. By symmetry, it lies on the \( y \)-axis, which is equivalent to \( x_c = 0 \), and it suffices to find \( y_c \).

\[ y_c = \frac{1}{M} \iint_D y \rho(x, y) \, dx \, dy, \quad M = \iint_D \rho(x, y) \, dx \, dy, \]

where \( D \) is the region occupied by lamina. Let us convert these integrals to polar coordinates.

\[ D = \{ 1 < r < 5, \ 0 < \theta < \pi \}, \]

because the first semicircle has radius 1, the second one has radius \( \sqrt{25} = 5 \), and \( D \) lies above the \( x \)-axis (this implies restrictions on \( \theta \)). Moreover,

\[ \rho = \frac{k}{r}, \quad dA = r \, dr \, d\theta, \quad y = r \sin \theta. \]

Therefore,

\[ M = \int_0^\pi \int_1^5 \frac{k}{r} r \, dr \, d\theta = k \int_1^5 dr \int_0^\pi d\theta = 4k\pi, \]

\[ \iint_D y \rho(x, y) \, dx \, dy = \int_1^5 \int_0^\pi r \sin \theta \frac{k}{r} r \, dr \, d\theta = k \int_0^\pi \sin \theta \, d\theta \int_1^5 r \, dr = 24k \]

(after calculating these integrals). Thus,

\[ y_c = \frac{24k}{4k\pi} = \frac{6}{\pi}. \]

**Problem 5.** The upper limit for \( r \) is always \( 2(1 + \cos \theta) \). The lower limit is \( 2 \cos \theta \), if \( \cos \theta > 0 \) (i.e. if \(-\pi/2 < \theta < \pi/2\)), and 0 otherwise (i.e. if \( \pi/2 \leq \theta \leq 3\pi/2 \)). So this area is

\[ \int_{-\pi/2}^{\pi/2} \int_{2 \cos \theta}^{2(1 + \cos \theta)} r \, dr \, d\theta + \int_{\pi/2}^{3\pi/2} \int_{0}^{2(1 + \cos \theta)} r \, dr \, d\theta. \]

Let us compute the first iterated integral. The inner integral is

\[ \int_{2 \cos \theta}^{2(1 + \cos \theta)} r \, dr = \frac{r^2}{2} \bigg|_{2 \cos \theta}^{2(1 + \cos \theta)} = \frac{1}{2} \left( 4(1 + \cos \theta)^2 - 4 \cos^2 \theta \right) = 2(1 + 2 \cos \theta) = 2 + 4 \cos \theta. \]

Therefore, the double integral is

\[ \int_{-\pi/2}^{\pi/2} (2 + 4 \cos \theta) \, d\theta = 2\theta + 4 \sin \theta \bigg|_{-\pi/2}^{\pi/2} = 2\pi + 8. \]
Let us compute the second iterated integral. The inner integral is

\[
\int_0^{2(1+\cos \theta)} r \, dr = \frac{(2(1 + \cos \theta))^2}{2} = 2(1 + \cos \theta)^2 = 2 \left( 1 + 2 \cos \theta + \cos^2 \theta \right) = 2 + 4 \cos \theta + 1 + \cos 2\theta = 3 + 4 \cos \theta + \cos 2\theta.
\]

So the double integral is

\[
\int_{\pi/2}^{3\pi/2} (3 + 4 \cos \theta + \cos 2\theta) d\theta = 3\theta + 4 \sin \theta + \frac{1}{2} \sin 2\theta \bigg|_{\pi/2}^{3\pi/2} = 3\pi - 8.
\]

Sum them up:

\[
(2\pi + 8) + (3\pi - 8) = 5\pi.
\]

This is the answer.

**Problem 6.** Since \((ye^{xy})_x = y^2e^{xy}\), we have by the Fundamental Theorem of Calculus:

\[
\int_0^y y^2e^{xy} \, dx = ye^{xy} \bigg|_{x=0}^{x=y} = ye^{y^2} - y.
\]

Hence by Fubini’s Theorem

\[
\int \int_D y^2e^{xy} \, dxdy = \int_0^3 \left[ \int_0^y y^2e^{xy} \, dx \right] dy = \int_0^3 (ye^{y^2} - y) \, dy = \int_0^3 ye^{y^2} \, dy - \int_0^3 y \, dy.
\]

To find the first integral, change variables: \(t = y^2\), \(0 \leq t \leq 9\), \(dt = 2ydy\).

\[
\int_0^3 ye^{y^2} \, dy = \int_0^9 e^t \frac{dt}{2} = \frac{1}{2} \left[ e^t \right]_{t=0}^{t=9} = \frac{e^9 - 1}{2}.
\]

And the second integral is much easier to compute:

\[
\int_0^3 y \, dy = \frac{y^2}{2} \bigg|_{y=0}^{y=3} = \frac{9}{2}.
\]

Thus, the answer is

\[
\frac{1}{2}(e^9 - 1) - \frac{9}{2} = \frac{e^9}{2} - 5.
\]