
12.1. Homogeneous Equations
Consider the equation
\[ y'' + \frac{1}{t} y' - \frac{1}{t^2} y = 0. \]
This is also a second-order homogeneous equation, but with non-constant coefficients. You can check that \( y_1(t) = t \) is its solution. Therefore, \( y(t) = C t \) is a solution for any real constant \( C \). Let us find other solutions. Employ variation of parameters: let \( C = C(t) \), then
\[
y(t) = C(t)t, \quad y' = C'(t)t + C(t), \quad y'' = C''(t)t + 2C'(t).
\]
Plug these into the equation:
\[
y'' + \frac{1}{t} y' - \frac{1}{t^2} y = \frac{C''(t)t + 2C'(t) + C(t) - C(t)}{t^2} = \frac{C''(t)t + 3C'(t)}{t^2} = 0.
\]
Letting \( z = C' \), we have:
\[
z'(t)t + 3z(t) = 0 \Rightarrow \frac{dz}{z} = -\frac{3}{t} \Rightarrow \log |z| = -3 \log |t| + K \Rightarrow |z| = e^K \frac{1}{|t|^3}.
\]
so
\[
z = \pm e^K \frac{1}{t^3} = C_0, \quad C_0 = \pm e^K \neq 0.
\]
The constant \( C_0 \) can take any nonzero value. But we lost the solution \( z = 0 \) when we divided by \( z \), and we get this solution when we let \( C_0 = 0 \). So the general solution is
\[
z = \frac{C_0}{t^3}, \quad C_0 \text{ is any real constant.}
\]
Then \( C = \int z(t)dt = -\frac{C_0}{2t^2} + C_1 \), and
\[
y(t) = C(t)t = -\frac{C_0}{2t} + C_1t = \boxed{C_1 t + \frac{C_2}{t}}
\]
Here, \( C_2 = -C_0/2 \). One solution is \( y_1 = t \), the other is \( y_2 = 1/t \). Any other solution is their linear combination. They form a fundamental set.

12.2. Initial Value Problem
Let us solve initial value problem: \( y(1) = 1, \ y'(1) = 0 \). Then
\[
y(t) = C_1 t + \frac{C_2}{t} \Rightarrow y(1) = C_1 + C_2 = 1,
\]
\[
y'(t) = C_1 - \frac{C_2}{t^2} \Rightarrow y'(1) = C_1 - C_2 = 0.
\]
Solving this system of equations, we get: \( C_1 = C_2 = 1/2 \). So the solution is
\[
y(t) = \boxed{\frac{1}{2} \left[ t + \frac{1}{t} \right]}.
\]
Consider a nonhomogeneous equation
\[ t^2 y'' - t(t + 2)y' + (t + 2)y = t^4. \]

The corresponding homogeneous equation
\[ t^2 y'' - t(t + 2)y' + (t + 2)y = 0 \]
has a solution \( y_1 = t \); so \( y = Ct \) is also a solution for any constant \( C \). Let us find a solution to the nonhomogeneous equation. Let \( C = C(t) \); then
\[ y = C(t)t, \quad y' = C'(t)t + C(t), \quad y'' = C''(t)t + 2C'(t), \]
plug this into the equation:
\[ t^2(C''(t)t + 2C'(t)) - t(t + 2)(C'(t)t + C(t)) + (t + 2)C(t)t = t^4. \]
Simplify:
\[ C'' - C' = t. \]
Let \( z = C' \), then \( z' - z = t \). Solving this equation, we get:
\[ z = -t - 1 + C_1 e^t. \]
So
\[ C = -\frac{1}{2} t^2 - t + C_1 e^t + C_2. \]
Thus, the answer is
\[ y = C(t)t = \left[ -\frac{1}{2} t^2 - t + C_1 e^t + C_2 \right] t = -\frac{1}{2} t^3 - t^2 + C_1 te^t + C_2 t. \]