14.1. Deduction of the Basic Equation

Assume we have a spring with constant $k$. On this spring, there is a mass $m$. There is a damping (resistance force) proportional to speed, with damping coefficient $\gamma$. Let $u(t)$ be the position of the mass; $u(t) = 0$ corresponds to the spring which is not extended or contracted. Then the spring force is $-ku(t)$, the damping force is $-\gamma u'(t)$, the gravity is $-mg$, so by Newton’s Second Law we have:

$$mu''(t) = -ku(t) - \gamma u'(t) - mg,$$

therefore,

$$mu'' + \gamma u' + ku = -mg.$$

Let $u = v - mg/k$, then $ku = kv - mg$, and $u' = u'$, $v'' = u''$. We have:

$$mv'' + \gamma v' + kv = 0.$$

14.2. Solving the Basic Equation

The characteristic equation is

$$m\lambda^2 + \gamma \lambda + k = 0 \Rightarrow \lambda_{1,2} = \frac{-\gamma \pm \sqrt{\gamma^2 - 4km}}{2m}.$$ 

There are three possible cases.

1. **Overdamping.** This is when $\gamma^2 - 4km > 0$. Then the roots $\lambda_{1,2}$ are real and distinct. The general solution is

$$v(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}.$$ 

The roots are negative, because $\gamma^2 - 4km < \gamma^2$ and $\sqrt{\gamma^2 - 4km} < \gamma$, so

$$\lambda_1 = \frac{-\gamma + \sqrt{\gamma^2 - 4km}}{2m} < 0, \quad \lambda_2 = \frac{-\gamma - \sqrt{\gamma^2 - 4km}}{2m} < 0.$$ 

So the solution tends to zero exponentially fast.

2. **Critical Damping.** This is when $\gamma^2 - 4km = 0$. The root is double: $\lambda = -\gamma/(2m)$. Therefore, the general solution is

$$v(t) = C_1 te^{-\gamma/2mt} + C_2 e^{-\gamma/2mt}.$$ 

This also tends to zero exponentially fast.

3. **Underdamping.** When $\gamma^2 - 4km < 0$. Then the roots are complex:

$$\lambda_{1,2} = \frac{-\gamma \pm i\sqrt{4km - \gamma^2}}{2m} = \rho \pm i\omega, \quad \rho = \frac{-\gamma}{2m}, \quad \omega = \frac{\sqrt{4km - \gamma^2}}{2m}.$$ 

So the solution is

$$v(t) = C_1 e^{\rho t} \cos(\omega t) + C_2 e^{\rho t} \sin(\omega t).$$ 

This also tends to zero, but it oscillates with quasi frequency $\omega$ and quasi period $2\pi/\omega$. The amplitude goes to zero: $\sqrt{C_1^2 + C_2^2} \rightarrow 0$.

Physical meaning: Damping wants to stop any movement. If it is small (underdamping), it cannot stop oscillations, but their amplitude goes to zero exponentially fast. If it is large (overdamping or critical damping), then it prevents oscillations, and the $v(t)$ just goes to zero exponentially fast.
14.3. Example
Let \( m = 1 \), \( \gamma = k = 2 \), and the initial conditions: \( v(0) = 0 \), \( v'(0) = 1 \). Then
\[
v'' + 2v' + 2v = 0.
\]
The characteristic equation is \( \lambda^2 + 2\lambda + 2 = 0 \) \( \Rightarrow \) \( \lambda_{1,2} = -1 \pm i \), and so the general solution is
\[
v(t) = C_1 e^{-t} \cos t + C_2 e^{-t} \sin t.
\]
The quasi frequency is one, and the quasi period is \( 2\pi \). Let us find \( C_1 \) and \( C_2 \). We have: \( v(0) = C_1 = 0 \), and so
\[
v(t) = C_2 e^{-t} \sin t \quad \Rightarrow \quad v'(t) = C_2 (e^{-t} \cos t - e^{-t} \sin t) \quad \Rightarrow \quad v'(0) = C_2 = 1.
\]
The answer is \( e^{-t} \sin t \). This oscillation has amplitude \( e^{-t} \).

By the way, what \( \gamma \) corresponds to the critical damping for \( m = 1 \), \( k = 2 \)? Answer: \( \gamma^2 - 4km = 0 \), \( \gamma^2 = 8 \), \( \gamma = 2\sqrt{2} \).

14.4. Writing Two Trig Functions as One
Assume we got:
\[
2e^{-t} \cos t - e^{-t} \sin t.
\]
Let us transform it. \( \sqrt{2^2 + (-1)^2} = \sqrt{5} \). So
\[
\sqrt{5}e^{-t} \left[ \frac{2}{\sqrt{5}} \cos t - \frac{1}{\sqrt{5}} \sin t \right].
\]
The point on the unit circle with coordinates \( (2/\sqrt{5}, -1/\sqrt{5}) \) corresponds to the angle \( \varphi = -\arcsin(1/\sqrt{5}) \).
Then we have:
\[
\frac{2}{\sqrt{5}} \cos t - \frac{1}{\sqrt{5}} \sin t = \cos t \cos \varphi - \sin t \sin \varphi = \cos(t - \varphi).
\]
The amplitude is \( \sqrt{5}e^{-t} \), and the phase (inside the cosine) is \( t - \varphi \). The initial phase (when \( t = 0 \)) is \( -\varphi = \arcsin(1/\sqrt{5}) \).