Lecture 4. Modeling with First-Order Equations II. July 1, 2013

4.1. Falling with Air Resistance

Assume a ball is falling in the air. Its mass is $m$, so the gravity is $mg$, where $g$ is the free fall acceleration. Let $v(t)$ be its speed at time $t$; then its acceleration is $v'(t)$. Assume the air resistance is $\alpha v(t)$, where $\alpha > 0$ is a certain constant. So it is proportional to the speed. Then, by Newton’s Second Law,

$$mv'(t) = mg - \alpha v(t).$$

Let us solve this equation. Separate variables:

$$\frac{dv}{dt} = v' = g - \frac{\alpha}{m} v \Rightarrow \int \frac{dv}{g - \frac{\alpha}{m} v} = dt \Rightarrow \int \frac{dw}{w} = -\frac{m}{\alpha} \log |w| = \frac{m}{\alpha} \log |g - (\alpha/m)v|.$$

Therefore,

$$\frac{m}{\alpha} \log |g - (\alpha/m)v| = t + C \Rightarrow \log |g - (\alpha/m)v| = \frac{\alpha}{m}(t + C) \Rightarrow |g - (\alpha/m)v| = e^{-\alpha C/m} e^{-\alpha t/m} \Rightarrow g - (\alpha/m)v = \pm e^{-\alpha C/m} e^{-\alpha t/m} = K e^{-\alpha t/m} \Rightarrow v = \frac{mg}{\alpha} - \frac{mK}{\alpha} e^{-\alpha t/m}.$$

Here, as usual, $K = \pm e^{-\alpha C/m}$ is any nonzero constant. Indeed, since $C$ is any real constant, $e^{-\alpha C/m}$ can assume any positive values, and $-e^{-\alpha C/m}$ can assume any negative values. But $K$ can also be zero. This value of $K$ corresponds to the solution $y = mg/\alpha$, which was lost when we divided by $g - (\alpha/m)v$. So $K$ is any real constant. We can denote $C_0 = -mK/\alpha$, simplifying the expression:

$$v(t) = \frac{mg}{\alpha} + C_0 e^{-\alpha t/m}.$$

Now, let us solve the following initial value problem. Assume the initial speed was zero: $v(0) = 0$. Then

$$\frac{mg}{\alpha} + C_0 = 0 \Rightarrow C_0 = -\frac{mg}{\alpha}.$$

Thus,

$$v(t) = \frac{mg}{\alpha} \left[1 - e^{-\alpha t/m}\right].$$

This function is increasing, but it is capped by the limiting value: $mg/\alpha$. So the speed does not increase up to infinity. This is the reason to use parachutes, because they increase the $\alpha$ coefficient and make the limiting speed $mg/\alpha$ smaller and safer for the person.

4.2. Concentration of a Pollutant

Assume you have a tank of capacity equal to 1000 gallons, and it is initially filled by a clean water. But there are the inflow and the outflow. The inflow: 10 gallons per minute, water with 10% salt. The outflow: 10 gallons per minute, well-stirred mixture. Assume $y(t)$ is how many gallons of salt are in the tank at time $t$. Then $y(0) = 0$. The inflow of salt is $10\% \cdot 10 = 1$ gallon per minute.
The outflow is \((y(t)/1000) \cdot 10 = y(t)/100\) gallons per minute. Therefore, the differential equation governing \(y\) is

\[y' = 1 - \frac{y}{100}, \quad y(0) = 0.\]

This is essentially the same equation as in the previous subsection, only with different coefficients. However, let us solve it using a different method: \textit{variation of parameters}. This is a linear inhomogeneous equation. Consider the corresponding homogeneous equation:

\[y' = -\frac{y}{100} \Rightarrow y(t) = Ce^{-t/100}.\]

Now, let \(C = C(t)\) be a function. Plug it into the original inhomogeneous equation, and get:

\[y' = C'(t)e^{-t/100} + C(t)\left(-\frac{t}{100}\right)e^{-t/100}, \quad 1 - \frac{y}{100} = 1 - \frac{C(t)}{100}e^{-t/100}.\]

They must be equal to each other, so

\[C'(t)e^{-t/100} = 1 \Rightarrow C'(t) = e^{t/100} \Rightarrow C(t) = 100e^{t/100} + K,\]

and plugging back into \(y(t)\), we have:

\[y(t) = \left[100e^{t/100} + K\right]e^{-t/100} = 100 + Ke^{-t/100}.\]

Now, since \(y(0) = 0\), we have: \(100 + K = 0 \Rightarrow K = -100\), and

\[y(t) = 100(1 - e^{-t/100}).\]

The limiting value is 100, which is 10% of the whole capacity. Which is natural: if the inflow is 10% salty water and the outflow is the well-stirred mixture, then eventually the concentration will be very close to 10%.