For each of the autonomous equations 1-4, do not solve them. Instead, analyze them qualitatively:
(a) draw graphs of solutions starting from different initial values;
(b) find when the solutions are increasing or decreasing;
(c) identify constant solutions, classify them: stable/unstable/semistable.

Problem 1. \( y' = y(y-1)(y-2)(y-3) \).

Solution. Let \( f(y) = y(y-1)(y-2)(y-3) \). Then
\[
\begin{align*}
  f(y) &= 0 \quad \Rightarrow \quad y = 0, 1, 2, 3,
\end{align*}
\]
so
\[
y(t) = 0, \quad y(t) = 1, \quad y(t) = 2, \quad y(t) = 3
\]
are equilibrium solutions. The function \( f \) has the following signs:
\[
\begin{cases}
  f(y) > 0, & y < 0; \\
  f(y) < 0, & 0 < y < 1; \\
  f(y) > 0, & 1 < y < 2; \\
  f(y) < 0, & 2 < y < 3; \\
  f(y) > 0, & y > 3.
\end{cases}
\]
So the solutions which start from \( y < 0 \), \( 1 < y < 2 \), \( y > 3 \) increase, while those which start from \( 0 < y < 1 \), \( 2 < y < 3 \) decrease. The equilibriums \( y = 0 \) and \( y = 2 \) are stable, and the equilibriums \( y = 1 \) and \( y = 3 \) are unstable.

Problem 2. \( y' = e^y - 1 \).

Solution. Let \( f(y) = e^y - 1 \). Then \( f(y) = 0 \quad \Rightarrow \quad y = 0 \), so \( y(t) = 0 \) is the only equilibrium solution. For \( y < 0 \), \( f(y) < 0 \), and for \( y > 0 \), \( f(y) > 0 \). So the solutions which start from \( y < 0 \) decrease, and which start from \( y > 0 \) increase. The equilibrium \( y = 0 \) is unstable.

Problem 3. \( y' = y^2(y-1)^2 \).

Solution. Let \( f(y) = y^2(y-1)^2 \). Then \( f(y) = 0 \quad \Rightarrow \quad y = 0, 1 \), and these are the equilibrium solutions. For all other \( y \), \( f(y) > 0 \), so all solutions other than equilibrium ones increase. Therefore, these two equilibrium solutions are semistable.

Problem 4. \( y' = ry \ln(K/y) \), where \( r, K > 0 \) are constants.

Solution. Note that in this case always \( y > 0 \), because logarithm is defined only for positive numbers. So
\[
f(y) := ry \log(K/y) = 0 \quad \Rightarrow \quad y = K.
\]
Moreover, \( f(y) > 0 \) for \( 0 < y < K \), and \( f(y) < 0 \) for \( y > K \). Therefore, the solution \( y(t) = K \) is a stable equilibrium. The solutions starting from \( 0 < y < K \) increase, and the solutions starting from \( y > K \) decrease.

**Problem 5.** For the initial value problem \( y' = t - y, \ y(0) = 1 \),

(i) find the exact solution and calculate \( y(1), y(2) \);
(ii) find approximately \( y(1), y(2) \) using Euler’s method for \( h = 1; \ h = 0.5; \)
(iii) find the relative errors (in percentage points) in these four results in (ii).

*Hint:* If the exact value is \( a \), and the approximate value is \( x \), then the relative error is \( \left| \frac{x - a}{a} \right| \).

**Solution.** (i) Let us find the general solution to this differential equation. This is a linear nonhomogeneous equation. First, let us solve the corresponding linear homogeneous equation:

\[
y' = -y \Rightarrow y = Ce^{-t}.
\]

Now, let us use variation of parameters: let \( C = C(t) \) and plug 
\[
y(t) = C(t)e^{-t}
\]
into the original nonhomogeneous equation. We have:

\[
y' = C'(t)e^{-t} - C(t)e^{-t} = t - C(t)e^{-t} \Rightarrow C'(t)e^{-t} = t \Rightarrow C'(t) = te^t \Rightarrow
\]

\[
C(t) = \int te^t dt = \int tde^t = te^t - \int e^t dt = te^t - e^t + K.
\]

Therefore,

\[
y(t) = C(t)e^{-t} = t - 1 + Ke^{-t}.
\]

Solve the initial value problem: \( y(0) = 1 \Rightarrow 0 - 1 + K = 1 \Rightarrow K = 2 \). The answer is

\[
y(t) = t - 1 + 2e^{-t}.
\]

Thus,

\[
y(0) = 2e^{-1} - 1
\]

(ii) For \( h = 1 \):

\[
y'(0) = 0 - y(0) = 0 - 1 = -1, \ y(1) = y(0) + 1 \cdot y'(0) = 0
\]

\[
y'(1) = 1 - y(1) = 1, \ y(2) = y(1) + 1 \cdot y'(1) = 1
\]

For \( h = 1/2 \):

\[
y'(0) = 0 - y(0) = -1, \ y(0.5) = y(0) + 0.5 \cdot y'(0) = 0.5;
\]

\[
y'(0.5) = 0.5 - y(0.5) = 0, \ y(1) = y(0.5) + 0.5 \cdot y'(0.5) = 0.25
\]

\[
y'(1) = 1 - y(1) = 0.5, \ y(1.5) = y(1) + 0.5 \cdot y'(1) = 0.75;
\]

\[
y'(1.5) = 1.5 - y(1.5) = 0.75, \ y(2) = y(1.5) + 0.5 \cdot y'(1.5) = 1.125
\]

(iii) For \( h = 1 \): \( y(1) \) has error 100\%, \( y(2) \) has error 21\%. For \( h = 0.5 \), \( y(1) \) has error 32\%, \( y(2) \) has error 11\%.
**Problem 6.** For the initial value problem \( y' = t/y, \ y(0) = 1 \),
(i) find the exact solution and calculate \( y(1) \);
(ii) find approximately \( y(1) \) using Euler’s method for \( h = 1; \ h = 0.5; \ h = 1/3; \)
(iii) find the relative errors (in percentage points) for the three results in (ii).

**Solution.** (i) \( y' = t/y \) \( \Rightarrow \) \( ydy = tdt \) \( \Rightarrow \) \( y^2/2 = t^2/2 + C \) \( \Rightarrow \) \( y^2 = t^2 + K \), where \( K = 2C \) is itself an arbitrary real constant; then
\[
y = \pm \sqrt{t^2 + K}.
\]
This is the general solution to the differential equation. Initial value problem: \( y(0) = 1 \) \( \Rightarrow \) \( K = 1 \) (and we need to take the plus sign), so
\[
y = \sqrt{t^2 + 1}.
\]
Therefore, \( y(1) = \sqrt{2} \).
(ii) For \( h = 1 \):
\[
y'(0) = 0/y(0) = 0, \ y(1) = y(0) + 1 \cdot y'(0) = 1
\]
For \( h = 0.5 \):
\[
y'(0) = 0/y(0) = 0, \ y(0.5) = y(0) + 0.5 \cdot y'(0) = 1;
y'(0.5) = 0.5/y(0.5) = 0.5, \ y(1) = y(0.5) + 0.5 \cdot y'(0.5) = 1.25
\]
For \( h = 1/3 \):
\[
y'(0) = 0, \ y(1/3) = y(0) + (1/3) \cdot y'(0) = 1;
y'(1/3) = 1/3 \cdot y(1/3) = 1/3, \ y(2/3) = y(1/3) + (1/3) \cdot y'(1/3) = 10/9;
y'(2/3) = 2/3 \cdot y(2/3) = 3/5, \ y(1) = y(2/3) + (1/3) \cdot y'(2/3) = 59/45
\]
(iii) For \( h = 1 \), the error is 29%. For \( h = 1/2 \), the error is 12%. For \( h = 1/3 \), the error is 7%.

**Problem 7.** Find the general solution of the Gompertz equation \( y' = ry \log(K/y) \). Also, find solutions of the following initial value problems for the Gompertz equation:
(i) \( y(0) = K/2 \);
(ii) \( y(0) = K \).

**Solution.** This is a separable equation (like any autonomous equation).
\[
\frac{dy}{y \log(K/y)} = rdt \ \Rightarrow \ \int \frac{dy}{y \log(K - \log y)} = rt + C.
\]
Let us calculate the integral in the right-hand side: change variables \( u = \log y \), then \( du = dy/y \), and
\[
\int \frac{dy}{y \log(K - \log y)} = \int \frac{du}{\log K - u} = - \log | \log K - u | = - \log | \log K - \log y |.
\]
Therefore,
\[
- \log | \log K - \log y | = rt + C \ \Rightarrow \ | \log K - \log y | = e^{-rt-C} \ \Rightarrow \ \log K - \log y = \pm e^{-C} e^{-rt} = De^{-rt}.
\]
Here, \( D = \pm e^{-C} \) is a constant which can take any nonzero values. Therefore,

\[
\log y = \log K - De^{-rt} \Rightarrow y = K \exp(-De^{-rt}).
\]

But we lost the constant solution \( y = K \), when we divided by \( y \log(K/y) \) (which is zero when \( y = K \)). This can be incorporated into the general formula, if we allow \( D = 0 \). Therefore, the general solution is

\[
y = K \exp(-De^{-rt})
\]

(i) \( y(0) = K/2 \) means that \( K \exp(-D) = K/2 \) \( \Rightarrow \ D = \ln 2 \). So

\[
y = K \exp(-\ln 2e^{-rt})
\]

(ii) \( y(t) = K \)

In problems 8 - 9, find the general solution of the differential equation.

**Problem 8.** \( y' = ty + e^{t^2/2} \).

**Solution.** This is a linear nonhomogeneous equation. First, let us solve the corresponding linear homogeneous equation:

\[
y' = ty \Rightarrow y = Ce^{t^2/2}.
\]

Now, let us use variation of parameters: let \( C = C(t) \) and plug \( y(t) = C(t)e^{t^2/2} \) into the original equation.

\[
y' = C'(t)e^{t^2/2} + C(t)te^{t^2/2} = tC(t)e^{t^2/2} + e^{t^2/2} \Rightarrow C'(t) = 1 \Rightarrow C(t) = t + K.
\]

Therefore,

\[
y = te^{t^2/2} + Ke^{t^2/2}
\]

**Problem 9.** \( y' = y^2 - y + 1 \).

**Solution.** This is a separable equation.

\[
\frac{dy}{y^2 - y + 1} = dt \Rightarrow \int \frac{dy}{y^2 - y + 1} = t + C.
\]

Let us calculate this integral:

\[
\int \frac{dy}{y^2 - y + 1} = \int \frac{dy}{(y - 1/2)^2 + (\sqrt{3}/2)^2} = \frac{1}{\sqrt{3}/2} \arctan \frac{y - 1/2}{\sqrt{3}/2}.
\]

Therefore,

\[
\frac{1}{\sqrt{3}/2} \arctan \frac{y - 1/2}{\sqrt{3}/2} = t + C \Rightarrow \tan \left( \frac{\sqrt{3}}{2}t + \frac{\sqrt{3}}{2}C \right) = \frac{y - 1/2}{\sqrt{3}/2}.
\]

Let \( K = \sqrt{3}C/2 \) be an arbitrary real constant. Then

\[
y = \frac{1}{2} + \frac{\sqrt{3}}{2} \tan \left( \frac{\sqrt{3}}{2}t + K \right)
\]
**Problem 10.** A pond of volume 1000 has inflow 1 per day of waste from a factory. The inflow has concentration of a pollutant \( p(t) = 1/2(1 - \cos(2\pi t)) \) at time \( t \). The meaning of this is at the beginning of the day, \( t = 0 \), we have \( p(0) = 0 \), so there is no pollutant; the waste is actually clean water. In the middle of the day, \( t = 1/2 \), the concentration equals \( p(1/2) = 1 \); that is, all of the waste consists of this pollutant. At the beginning of the next day, \( p(1) = 0 \), so there is again no pollutant.

The outflow is also 1 per day (a homogeneous mixture of water and the pollutant). Originally (at time \( t = 0 \)), the pond was clean (no pollutant.) What is the concentration of a pollutant at time \( t = 1 \) (the next day)?

**Solution.** Let \( y(t) \) be the amount of pollutant at time \( t \). During the time interval \([t, t + dt]\), we have: the inflow of waste is \( dt \), and the inflow of the pollutant is \( p(t)dt \). The outflow of waste is \( dt \), and it contains \( y(t)dt/1000 \) of the pollutant. Therefore,

\[
y(t + dt) = y(t) + p(t)dt - \frac{y(t)dt}{1000} \Rightarrow \frac{y(t + dt) - y(t)}{dt} = p(t) - \frac{y(t)}{1000}.
\]

Therefore,

\[
y' = p(t) - \frac{1}{1000}y.
\]

Let us solve this linear nonhomogeneous equation. First, consider the corresponding linear homogeneous equation.

\[
y' = -\frac{1}{1000}y \Rightarrow y = Ce^{-t/1000}.
\]

Variation of parameters: let \( C = C(t) \) and plug \( y = C(t)e^{-t/1000} \) into the nonhomogeneous equation.

\[
C'e^{-t/1000} - \frac{1}{1000}Ce^{-t/1000} = p(t) - \frac{1}{1000}Ce^{-t/1000} \Rightarrow C' = e^{t/1000}p(t).
\]

Therefore,

\[
C(t) = \int \frac{1}{2}(1 - \cos(2\pi t))e^{t/1000}dt = \frac{1}{2} \int e^{t/1000}dt - \frac{1}{2} \int \cos(2\pi t)e^{t/1000}dt.
\]

It is known (see any integral table) that

\[
\int \cos(ax)e^{bx}dx = \frac{1}{a^2 + b^2}e^{bx}(a \sin ax + b \cos ax).
\]

Therefore, this integral is equal to

\[
C(t) = \frac{1000}{2}e^{t/1000} - \frac{1}{2} (2\pi)^2 + (1/1000)^2 e^{t/1000} \left( 2\pi \sin(2\pi t) + \frac{1}{1000} \cos(2\pi t) \right) + K.
\]

Finally,

\[
y(t) = C(t)e^{-t/1000} = 500 - \frac{1}{2((2\pi)^2 + (1/1000)^2) e^{t/1000}} \left( 2\pi \sin(2\pi t) + \frac{1}{1000} \cos(2\pi t) \right) + Ke^{-t/1000}.
\]
This is the general solution to the differential equation. Now, \( y(0) = 0 \), because initially the pond is clean. Therefore,

\[
y(0) = 0 = 500 - \frac{1}{2((2\pi)^2 + (1/1000)^2)} \frac{1}{1000} + K \Rightarrow K = -500 + \frac{1}{2((2\pi)^2 + (1/1000)^2)} \frac{1}{1000}.
\]

Thus,

\[
y(1) = 500 - \frac{1}{2((2\pi)^2 + (1/1000)^2)} \frac{1}{1000} + Ke^{-1/1000} = \frac{K(e^{-1/1000} - 1)}{1000} = 0.49975
\]

The concentration is \( y(1)/1000 = 0.00049975 \).

**Problem 11.** We have a mass of bacteria, which originally weighs 10lb. (Instead of a quantity of bacteria, we measure its mass.) The rate of increase of the quantity of bacteria is proportional to the square of the current quantity, because every pair of bacteria provide a new one. The proportional coefficient is 0.1, that is, the rate of increase it 0.1 times the square of the current mass. Determine the moment when the population will explode: become infinite.

**Solution.** Let \( y(t) \) be the quantity of bacteria. Then

\[
y' = 0.1y^2, \quad y(0) = 10.
\]

Separate variables:

\[
\frac{dy}{y^2} = 0.1 dt \Rightarrow \int \frac{dy}{y^2} = 0.1t + C \Rightarrow -\frac{1}{y} = 0.1t + C \Rightarrow y = \frac{1}{-C - 0.1t}.
\]

From \( y(0) = 10 \) we find \( C = -0.1 \). Therefore,

\[
y = \frac{10}{1 - 0.1t} = \frac{10}{1 - t}.
\]

This becomes infinite at \( t = 1 \).

**Problem 12.** In actuarial models, they study \( p(x) \), which is the probability that a man will be alive at age \( x \). This is called the survival function. The function

\[
\mu(x) = -\frac{p'(x)}{p(x)}
\]

is called the force of mortality. Makeham (1860) proposed the model when

\[
\mu(x) = A + Be^{\alpha x},
\]

where \( A, B, \alpha > 0 \) are constants. The first term \( A \) takes into account accidents, which are independent of age. The second term is related to health. Find \( p(x) \) in this model, using the initial condition \( p(0) = 1 \) (at age zero, everybody is alive by definition).

**Solution.** We have:

\[
p(x) = C \exp \left(-\int \mu(x) dx\right) = C \exp \left(-Ax - \frac{B}{\alpha} e^{\alpha x}\right).
\]
Frm $p(x) = 0$ we get: $C \exp(-B/\alpha) = 1 \Rightarrow C = \exp(B/\alpha)$. Therefore,

$$p(x) = \exp\left(-Ax - \frac{B}{\alpha} \left(e^{\alpha x} - 1\right)\right)$$

**Problem 13.** (Harvesting a renewable resource.) Suppose the population $y(t)$ of fish in a given area of ocean is given by

$$y' = r(1 - y/K)y.$$  

The rate at which the fish are caught is proportional to its population:

$$y' = r(1 - y/K)y - Ey.$$  

This is *the Schaefer model.*

(i) Show that $E < r$, then there are two equilibrium points, 

$$y_1 = 0 \quad \text{and} \quad y_2 = K(1 - E/r) > 0.$$  

Show that the point $y_1$ is unstable and $y_2$ is stable.

(ii) A sustainable yield $Y$ of the fishery is a rate $Y = Ey_2$ at which fish can be caught indefinitely. So $Y$ is a function of $E$; find $E$ so that $Y$ is maximal.

**Solution.** (i) Solve for  

$$f(y) = r(1 - y/K)y - Ey = 0 :$$  

we have: either $y = 0$ or $r(1 - y/K) - E = 0 \Rightarrow y = y_2$. We have:

$$f(y) = (r - E)y - \frac{r}{K}y^2 = y\left((r - E) - \frac{r}{K}y\right) = y(y - y_2).$$  

Therefore,

$$\begin{cases}  
  f(y) < 0, & y < 0; \\
  f(y) > 0, & 0 < y < y_2; \\
  f(y) < 0, & y > y_2  
\end{cases}$$  

The solutions starting from below 0 or above $y_2$ decrease, and those starting in between 0 and $y_2$ decrease. Therefore, $y_1 = 0$ is unstable and $y_2$ is stable.

(ii) $Y = EK(1 - E/r) = KE - (K/r)E^2$. Therefore, $Y'(E) = K - 2EK/r = 0 \Rightarrow E = \sqrt{r/2}$
Problem 1

Problem 2

Problem 3

Problem 4