In Problems 1-3, find the general solution of the equation.

**Problem 1.** \( y'' - y = e^{2t} \).

**Solution.** First, let us solve the corresponding homogeneous equation: \( y'' - y = 0 \). Characteristic equation: \( \lambda^2 - 1 = 0 \), with roots \( \lambda = \pm 1 \). The general solution of the homogeneous equation is

\[ C_1 e^t + C_2 e^{-t}. \]

Then, let us find a particular solution for the nonhomogeneous equation. Since the exponent 2 does not coincide with the roots of the characteristic equation, let \( y = Ae^{2t} \). Then

\[ y'' - y = 4Ae^{2t} - Ae^{2t} = 3Ae^{2t} = e^{2t}. \]

Comparing coefficients, we get: \( 3A = 1 \), so \( A = 1/3 \). Answer:

\[ \frac{1}{3} e^{2t} + C_1 e^t + C_2 e^{-t} \]

**Problem 2.** \( y'' - y = te^{2t} + 2e^t + t \).

**Solution.** The general solution of the homogeneous equation is already found in the previous problem: \( C_1 e^t + C_2 e^{-t} \). Now, let us find a particular solution of the nonhomogeneous equation. First, let us find \( y_1 \) such that

\[ y''_1 - y_1 = te^{2t}. \]

Since 2 in the exponent does not coincide with the roots of the characteristic equation, we do not need to raise the degree of the polynomial \( t \). So try

\[ y_1 = (At + B)e^{2t}. \]

Then

\[ y'_1 = Ae^{2t} + 2(At + B)e^{2t}, \quad y''_1 = 2Ae^{2t} + 2Ae^{2t} + 4(At + B)e^{2t}. \]

Therefore,

\[ y''_1 - y_1 = 3Ate^{2t} + (4A + 3B)e^{2t}. \]

This must be equal to \( te^{2t} \). Comparing coefficients, we get:

\[ 3A = 1, \quad 4A + 3B = 0. \]

Solve for \( A \) and \( B \):

\[ A = \frac{1}{3}, \quad B = -\frac{4}{9}. \]

Therefore,

\[ y_1 = \frac{1}{3} te^{2t} - \frac{4}{9} e^{2t}. \]
Now, let us look for $y_2$ such that
\[ y''_2 - y_2 = 2e^t. \]
Since the 1 in the exponent coincides with a root of the characteristic equation, we need to raise the degree of polynomial 2 from zero to one. Try
\[ y_2 = Ate^t. \]
Then
\[ y'_2 = Ae^t + Ate^t, \quad y''_2 = Ae^t + Ae^t + Ate^t. \]
Therefore,
\[ y''_2 - y_2 = 2Ae^t = 2e^t, \]
so $A = 1$, and $y_2 = te^t$.

Finally, find $y_3$ such that
\[ y''_3 - y_3 = te^{0t}. \]
The 0 in the exponent is not a root of the characteristic equation, so we do not need to raise the degree of the polynomial $t$: try
\[ y_3 = (At + B)e^{0t} = At + B. \]
Then $y''_3 = 0$, and $y''_3 - y_3 = -At - B = t$. So $A = -1, B = 0$. We have: $y_3 = -t$.

Finally, a particular solution to the original equation is
\[ y_1 + y_2 + y_3 = \frac{1}{3}te^{2t} - \frac{4}{9}e^{2t} + te^t - t, \]
and the general solution is
\[ \frac{1}{3}te^{2t} - \frac{4}{9}e^{2t} + te^t - t + C_1e^t + C_2e^{-t}. \]

**Problem 3.** $y'' + 2y' + 8y = -e^t$.

**Solution.** Solve the homogeneous equation. Characteristic equation: $\lambda^2 + 2\lambda + 8 = 0$, with roots $\lambda = -1 \pm \sqrt{7}i$. So the general solution of the homogeneous equation is
\[ C_1e^{-t}\cos(\sqrt{7}t) + C_2e^{-t}\sin(\sqrt{7}t). \]
Now, let us find a particular solution of the nonhomogeneous equation. Since 1 in the $e^t$ does not coincide with $-1 \pm \sqrt{7}i$, then try
\[ y = Ae^t. \]
Then
\[ y'' + 2y' + 8y = 11Ae^t = -e^t, \]
so $A = -1/11$. Therefore,
\[ y = -\frac{1}{11}e^t \]
is a particular solution of the nonhomogeneous equation, and

\[ y = -\frac{1}{11} e^t + C_1 e^{-t} \cos(\sqrt{7}t) + C_2 e^{-t} \sin(\sqrt{7}t) \]

is the general solution of this equation.

In Problems 4-9, write the general solution of the equation with undetermined coefficients (but do not find the coefficients). Indicate which coefficients are undetermined but fixed (such as \(A, B\)), and which are general constants which are to be determined for the initial value problem \((C_1, C_2)\).

Example:

\[ y'' - y = 2te^t \quad \Rightarrow \quad y = C_1 e^t + C_2 e^{-t} + (At^2 + Bt)e^t. \]

**Problem 4.** \( y'' - y' - 6y = t^2 e^{-t}. \)

**Solution.** \( \lambda^2 - \lambda - 6 = 0 \quad \Rightarrow \lambda_{1,2} = 3, -2. \) So \( y_0 = C_1 e^{-2t} + C_2 e^{3t}. \) Now, \( y_1 = (At^2 + Bt + C)e^{-t}. \)

**Problem 5.** \( y'' - 4y = -0.5t^3 e^{3t} + 1. \)

**Solution.** \( \lambda^2 - 4 = 0 \quad \Rightarrow \lambda = \pm 2, \) and \( y_0 = C_1 \cos(2t) + C_2 \sin(2t). \) Now, \( y_1 = (At^3 + Bt^2 + Ct + D)e^{3t} + E. \)

**Problem 6.** \( y'' - 4y' + 3y = -4t^2 e^t + e^{3t}. \)

**Solution.** \( \lambda^2 - 4\lambda + 3 = 0 \quad \Rightarrow \lambda = 1, 3. \) So \( y_0 = C_1 e^t + C_2 e^{3t}. \) Now, \( y_1 = (At^3 + Bt^2 + Ct + D)te^{3t}. \)

**Problem 7.** \( y'' + 4y = -3(t^5 - 4t)e^{-t}. \)

**Solution.** \( \lambda^2 + 4 = 0 \quad \Rightarrow \lambda = \pm 2i, \) so \( y_0 = C_1 \cos(2t) + C_2 \sin(2t). \) And \( y_1 = (At^5 + Bt^4 + Ct^3 + Dt^2 + Et + F)e^{-t}. \)

**Problem 8.** \( y'' - 6y' + 9y = -(t^2 + t)e^{3t} - 4t^3 + t^2 e^t. \)

**Solution.** \( \lambda^2 - 6\lambda + 9 = 0 \quad \Rightarrow \lambda = 3. \) So \( y_0 = C_1 e^{3t} + C_2 te^{3t}. \) Now, \( y_1 = (At^4 + Bt^3 + Ct^2)e^{3t} + Dt^3 + Et^2 + Ft + G + (Ht^2 + It + J)e^t. \)

**Problem 9.** \( y'' + 3y' + 2y = -2t^2 + 3 + (t^3 - 4t)e^{-t}. \)

**Solution.** \( \lambda^2 + 3\lambda + 2 = 0 \quad \Rightarrow \lambda = -1, -2. \) Now, \( y_0 = C_1 e^{-t} + C_2 e^{-2t}, \) and \( y_1 = At^2 + Bt + C + (Dt^4 + Et^3 + Ft^2 + Gt)e^{-t}. \)