Problem 1. Determine the steady-state solution for mechanical vibrations with 
\[ m = 1, \ k = 2, \ \gamma = 1, \ F_0(t) = 3 \cos t. \]

Solution. We have:
\[ mu'' + \gamma u' + ku = 3 \cos t \Rightarrow u'' + u' + 2u = 3 \cos t. \]
Let us find a particular solution of the nonhomogeneous equation in the form 
\[ u_1 = A \cos t + B \sin t. \]
This will be the steady-state solution. After plugging in the equation, we get:
\[ A = \frac{3}{2}, \ B = \frac{3}{2}. \]

Problem 2. For a mechanical system with no damping, \( k = 2, \ m = 8, \) and the external force 
\( F_0(t) = -2 \sin(\omega t), \) find the value of \( \omega \) which causes the resonance. Find \( u(t) \) for this \( \omega, \) if the initial position \( u(0) = 1, \) and the initial speed is zero.

Solution. \( 8u'' + 2u = -2 \sin(\omega t). \) The homogeneous equation \( 8u'' + 2u = 0 \) has characteristic equation \( 8\lambda^2 + 2 = 0 \Rightarrow \lambda = \pm (1/2)i. \) So the general solution of the homogeneous equation is
\[ u = C_1 \cos(t/2) + C_2 \sin(t/2), \]
and the internal frequency (frequency in absence of external forces, which corresponds to zero right-hand side) is equal to \( 1/2. \) The external frequency is \( \omega, \) so the resonance happens when \( \omega = 1/2. \) For this \( \omega, \) we should find a particular solution to the nonhomogeneous equation in the form
\[ u = At \cos(t/2) + Bt \sin(t/2). \]
After calculation, we get: \( A = 1/4, \ B = 0. \) So the general solution to the nonhomogeneous equation is
\[ u = \frac{1}{4}t \cos(t/2) + C_1 \cos(t/2) + C_2 \sin(t/2). \]
From the initial conditions \( u(0) = 1 \) and \( u'(0) = 0, \) we find \( C_1 \) and \( C_2: \ C_1 = 1, \) and \( C_2 = -1/2. \)

Problem 3. Let \( m = 1, \ k = 4. \) For which \( \gamma \) is there an overdamping? underdamping?

Solution. \( \gamma > 2\sqrt{km} = 4: \) overdamping. \( \gamma < 4: \) underdamping.

Problem 4. Consider a mechanical system without external force, with parameters 
\[ m = 1, \ k = 2, \ \gamma = 1, \ u(0) = 1, \ u'(0) = -1. \]
Find the amplitude (dependent on \( t), \) and initial phase.
Solution. $u'' + u' + 2u = 0$ has characteristic equation
$$\lambda^2 + \lambda + 2 = 0 \Rightarrow \lambda = \frac{-1 \pm \sqrt{1^2 - 4 \cdot 1 \cdot 2}}{2} = \frac{-1 \pm \sqrt{7}}{2}.$$ The general solution is
$$u = C_1 e^{-t/2} \cos\left(\frac{\sqrt{7}}{2} t\right) + C_2 e^{-t/2} \sin\left(\frac{\sqrt{7}}{2} t\right).$$ From the initial conditions, we get $C_1, C_2$: $C_1 = u(0) = 1$, and $C_2 = -1/\sqrt{7}$. Consider coefficients $K_1(t) = C_1 e^{-t/2} = e^{-t/2}$, $K_2(t) = C_2 e^{-t/2} = -\frac{1}{\sqrt{7}} e^{-t/2}$.

The amplitude is
$$A(t) = \sqrt{K_1^2(t) + K_2^2(t)} = \sqrt{1 + 1/7} e^{-t/2} = \sqrt{8/7} e^{-t/2}.$$ The initial phase is $\varphi$ such that
$$\cos \varphi = K_2(0)/A(0) = \frac{-1/\sqrt{7}}{\sqrt{8/7}} = -\frac{1}{2\sqrt{2}}, \quad \sin \varphi = K_1(0)/A(0) = \frac{1}{\sqrt{8/7}} = \sqrt{7}/8.$$ This angle is $\pi - \sin^{-1}(\sqrt{7}/8)$, because it lies in the second quadrant.

Problem 5. A circuit has a capacitor of 1, a resistor of 3, and an inductor of 2. The initial charge of the capacitor is $-2$ and there is no initial current. The battery gives $E(t) = 3 \cos(2t)$. Find the charge $Q$ on the capacitor at any time $t$. What is the steady-state solution?

Solution. $C = 1, R = 3, L = 2, Q(0) = -2, Q'(0) = I(0) = 0$. So
$$2Q'' + 3Q' + Q = 3 \cos(2t).$$ Homogeneous equation:
$$2\lambda^2 + 3\lambda + 1 = 0, \quad \lambda = -1, -\frac{1}{2}.$$ The general solution to the homogeneous equation:
$$Q_0(t) = C_1 e^{-t} + C_2 e^{-t/2}.$$ Find a particular solution to the nonhomogeneous equation in the form of
$$Q_1(t) = A \cos(2t) + B \sin(2t).$$ Plugging in the equation, we get: $B = 18/85, A = -21/85$. This $Q_1$ is the steady-state solution. From the initial conditions, we get: $C_1 = 221/85$, and $C_2 = -74/17$.

Problem 6. A circuit has a capacitor $C$, an inductor $L$, and no resistor. The parameters $C$ and $L$ are given. The battery gives you $E(t) = E_0 \cos(\omega t)$. Find $\omega$ such that there is a resonance.

Solution. $LQ'' + Q/C = E_0 \cos(\omega t)$. The homogeneous equation is
$$LQ'' + \frac{1}{C} Q = 0.$$
The characteristic equation is
\[ L\lambda^2 + \frac{1}{C} = 0 \Rightarrow \lambda^2 = -\frac{1}{LC} \Rightarrow \lambda = \pm \frac{1}{\sqrt{CL}}i. \]
Therefore, the internal frequency is \(1/\sqrt{CL}\). So the resonance is when \(\omega = 1/\sqrt{CL}\).

**Problem 7.** A circuit has \(R = 1\), \(L = 1\), and no capacitor. The battery gives the constant voltage: \(E(t) = 2\). Suppose initially there were no charge and no current in the system. Find the current at time \(t\). What is the limit of the current as \(t \to \infty\)?

**Solution.** \(Q'' + Q' = 2, Q(0) = Q'(0) = 0\). Homogeneous equation: \(Q'' + Q' = 0\). The characteristic equation: \(\lambda^2 + \lambda = 0 \Rightarrow \lambda = 0, -1\). So the general solution of the homogeneous equation is
\[ Q_0(t) = C_1e^{-t} + C_2. \]
A particular solution to the nonhomogeneous equation is
\[ Q_1(t) = At, \]
because the right-hand side 2 corresponds to \(e^{0t}\), but \(\lambda = 0\) is a root of the characteristic equation, so we have to raise the degree of the polynomial from zero to one. You will find \(A = 2\). So the general solution of the original nonhomogeneous equation is
\[ Q(t) = Q_0(t) + Q_1(t) = C_1e^{-t} + C_2 + 2t. \]
From the initial conditions, we get: \(C_1 = 2\) and \(C_2 = -2\). So
\[ Q(t) = 2e^{-t} - 2 + 2t \Rightarrow I(t) = Q'(t) = -2e^{-t} + 2 \to 2 \text{ as } t \to \infty. \]

**Problem 8.** A circuit has \(C = 2\), \(L = 4\), and no resistor. There is no battery. The initial charge of the capacitor is \(-1/2\), and the initial current is \(-1\). Find the amplitude, the phase and the initial phase.

**Solution.**
\[ Q/2 + 4Q'' = 0 \Rightarrow Q'' + \frac{1}{8}Q = 0, \quad Q(0) = -1/2, \quad Q'(0) = -1. \]
Therefore,
\[ Q(t) = C_1\cos(t/\sqrt{8}) + C_2\sin(t/\sqrt{8}). \]
From the initial conditions, we find: \(C_1 = -1/2\), \(C_2 = -\sqrt{8}\). So
\[ Q(x) = -\sqrt{8}\sin(t/\sqrt{8}) - \frac{1}{2}\cos(t/\sqrt{8}). \]
Amplitude: \(A = \sqrt{C_1^2 + C_2^2} = \sqrt{33}/4\). Initial phase:
\[ \cos \varphi = \frac{C_2}{A} = \frac{-\sqrt{8}}{\sqrt{33}/4} = -\sqrt{\frac{32}{33}}, \quad \sin \varphi = \frac{C_1}{A} = \frac{-1/2}{\sqrt{33}/4} = -\frac{1}{\sqrt{33}}. \]
Therefore,
\[ \varphi = \pi + \sin^{-1}\left(\frac{1}{\sqrt{33}}\right). \]
Phase: \(\varphi + t/\sqrt{8}\).