
Try to solve the equation
\[ ay'' + by' + cy = 0, \]
where \(a, b, c\) are real numbers. Plug in \(y = e^{\lambda t}\), and get (after canceling out \(e^{\lambda t}\)):
\[ a\lambda^2 + b\lambda + c = 0. \]

If \(b^2 - 4ac > 0\), then this equation has two distinct real roots
\[ \lambda_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]
and the general solution is
\[ y = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}. \]

If \(b^2 - 4ac < 0\), then the roots are complex, but we can use this formula too. We need to make sense of \(e^z\) with complex \(z\). This is done as follows (we shall talk later about why this is true): for real \(t\),
\[ e^{it} = \cos t + i \sin t. \]

This is Euler’s formula. And so \(e^{a+bi} = e^a (\cos b + i \sin b)\).

**Example.** \(y'' - 4y' + 6y = 0\). The characteristic equation
\[ \lambda^2 - 4\lambda + 6 = 0 \Rightarrow \lambda = \frac{4 \pm \sqrt{4^2 - 6 \cdot 4}}{2} = \frac{4 \pm \sqrt{-8}}{2} = 2 \pm \sqrt{2}i. \]

So the general solution is
\[ C_1 e^{(2+\sqrt{2})t} + C_2 e^{(2-\sqrt{2})t} = C_1 e^{2t} \cos (\sqrt{2}t) + iC_1 e^{2t} \sin (\sqrt{2}t) + C_2 e^{2t} \cos (\sqrt{2}t) - iC_2 e^{2t} \sin (\sqrt{2}t) \]
\[ = (C_1 + C_2) e^{2t} \cos (\sqrt{2}t) + i(C_1 - C_2) e^{2t} \sin (\sqrt{2}t). \]

We denote \(C_1 + C_2 = K_1\), and \(i(C_1 - C_2) = K_2\). Since \(C_1, C_2\) are arbitrary real constants, \(K_1\) and \(K_2\) are also arbitrary real constants, because for any given \(K_1\) and \(K_2\) we can solve for \(C_1\) and \(C_2\). So we can write the general solution as
\[ K_1 e^{2t} \cos (\sqrt{2}t) + K_2 e^{2t} \sin (\sqrt{2}t). \]

In general, if we have \(\lambda_{1,2} = \mu \pm i\omega\), then the general solution is
\[ y = K_1 e^{\mu t} \cos (\omega t) + K_2 e^{\mu t} \sin (\omega t). \]