
Example 1. Consider the equation
\[ y'' + y' - 2y = e^{2t}. \]
This is a second-order equation, because it contains the unknown function \( y \) and its first and second derivatives. This is a linear equation, because the left-hand side linearly depends on \( y \): this is a linear combination of \( y'' \), \( y' \) and \( y \). And it is nonhomogeneous, because the right-hand side (the free term) is nonzero.

First, solve the corresponding homogeneous equation:
\[ y'' + y' - 2y = 0. \]
Characteristic equation: \( \lambda^2 + \lambda - 2 = 0 \), and the roots are 1 and \( -2 \). Therefore, the general solution of the homogeneous equation is
\[ y_0(t) = C_1 e^t + C_2 e^{-2t}. \]

**General Rule.** General solution \( y_0 \) of the homogeneous equation + particular solution \( y_1 \) of the nonhomogeneous equation = general solution of the nonhomogeneous equation.

Indeed, if
\[ y_0'' + y_0' - 2y_0 = 0, \quad \text{and} \quad y_1'' + y_1' - 2y_1 = e^{2t}, \]
then for \( y = y_0 + y_1 \)
\[ y'' + y' - 2y = (y_0 + y_1)'' + (y_0 + y_1)' - 2(y_0 + y_1) = y_0'' + y_0' - 2y_0 + (y_1'' + y_1' - 2y_1) = e^{2t} + 0 = e^{2t}. \]

So to find a general solution of a nonhomogeneous equation, you need to (1) find general solution of the corresponding homogeneous equation, and (2) find any particular solution of the nonhomogeneous equation.

We already did (1). Let us do (2). Try \( y_1 = Ae^{2t} \), with unknown coefficient \( A \).
\[ y' = 2Ae^{2t}, \quad y'' = 4Ae^{2t} \quad \Rightarrow \quad y'' + y' - 2y = 4Ae^{2t} + 2Ae^{2t} - 2Ae^{2t} = 4Ae^{2t}, \]
which must be equal to \( e^{2t} \). So \( 4A = 1 \), and \( A = 1/4 \). Therefore,
\[ y_1 = \frac{1}{4} e^{2t}, \]
and
\[ y = y_0 + y_1 = \frac{1}{4} e^{2t} + C_1 e^t + C_2 e^{-2t} \]
is a general solution of the original nonhomogeneous equation.

Example 2. \( y'' + y' - 2y = (t + 2)e^{2t} \). We already found \( y_0 = C_1 e^t + C_2 e^{-2t} \); try
\[ y_1 = (At + B)e^{2t}. \]
Then
\[ y' = 2(At + B)e^{2t} + Ae^{2t} = (2At + 2B + A)e^{2t}, \]
\[ y'' = 4(At + B)e^{2t} + 2Ae^{2t} + 2Ae^{2t} = (4At + 4B + 4A)e^{2t}, \]
and
\[ y'' + y' - 2y = (4At + 4B + 4A)e^{2t} + (2At + 2B + A)e^{2t} - 2(At + B)e^{2t} = 4At e^{2t} + (5A + 4B)e^{2t}. \]
Comparing coefficients, we get:
\[ 4A = 1, \quad 5A + 4B = 2 \quad \Rightarrow \quad A = \frac{1}{4}, \quad B = \frac{3}{16}. \]
Therefore,
\[ \frac{1}{4}te^{2t} + \frac{3}{16}e^{2t} \]
is a particular solution of the nonhomogeneous equation, and
\[ \frac{1}{4}te^{2t} + \frac{3}{16}e^{2t} + C_1 e^t + C_2 e^{-2t} \]
is the general solution.

**General Rule.** If you have right-hand side in the form: polynomial of degree \( n \) times \( e^{kt} \), that is,
\[ (a_n t^n + a_{n-1} t^{n-1} + \ldots + a_1 t + a_0) e^{kt}, \]
where \( k \) is not a root of the characteristic equation, then find \( y_1 \) in the form of another polynomial of degree \( n \) (with unknown coefficients) times \( e^{at} \). Namely:
\[ (A_n t^n + A_{n-1} t^{n-1} + \ldots + A_1 t + A_0) e^{kt}, \]
where \( A_n, \ldots, A_0 \) are to be determined.
For example, if the right-hand side is \( t^3 e^{2t} \), then try \( y_1 = (At^3 + Bt^2 + Ct + D)e^{2t} \).