Lecture 17. Mechanical Vibrations. February 13, 2015

Consider the following model. A body of mass $m$ is moving horizontally on a spring. It slides on a smooth horizontal surface. The position of the body is $u$ (where 0 is the equilibrium). Two forces act upon this body:

- the spring force, which wants to return this body to equilibrium: $F_1 = -ku$, proportional to the distance from the equilibrium. $k > 0$ is a coefficient. The minus sign stands because the force wants to pull the body back to the equilibrium;
- damping: air or water resistance, proportional to the speed: $F_2 = -\gamma u'$, where $\gamma > 0$. The minus sign indicates that the force wants to slow it down.

By Newton’s Second Law, mass times acceleration equals the sum of all forces acting upon this body:

$$mu'' = F_1 + F_2 = -ku - \gamma u'.$$

Therefore,

$$mu'' + \gamma u' + ku = 0.$$

If there is no damping, then we have:

$$mu'' + ku = 0.$$

Let us solve this:

$$u'' + \frac{k}{m}u = 0, \quad u'' + \omega^2 u = 0,$$

where $\omega = \sqrt{k/m}$. Solve this equation: the characteristic equation is $\lambda^2 + \omega^2 = 0 \Rightarrow \lambda = \pm i\omega$. So the general solution is

$$u(t) = C_1 \cos(\omega t) + C_2 \sin(\omega t).$$

This is called harmonic oscillator. It is a very important basic model for physics and engineering. This function $u(t)$ is periodic, with the period

$$T = \frac{2\pi}{\omega}.$$

Which means that for every $t$, we have $u(t + T) = u(t)$. Why is that? Indeed, for every $t$ we have:

$$C_1 \cos \left( \omega \left( t + \frac{2\pi}{\omega} \right) \right) + C_2 \sin \left( \omega \left( t + \frac{2\pi}{\omega} \right) \right) = C_1 \cos(\omega t + 2\pi) + C_2 \sin(\omega t + 2\pi) = C_1 \cos(\omega t) + C_2 \sin(\omega t).$$

We can write the sum of these trig functions as one trig function, as follows:

$$C_1 \cos(\omega t) + C_2 \sin(\omega t) = \sqrt{C_1^2 + C_2^2} \left( \frac{C_1}{\sqrt{C_1^2 + C_2^2}} \cos(\omega t) + \frac{C_2}{\sqrt{C_1^2 + C_2^2}} \sin(\omega t) \right).$$

Now, let us find angle $\varphi$ such that

$$\sin \varphi = y = \frac{C_1}{\sqrt{C_1^2 + C_2^2}}, \quad \cos \varphi = x = \frac{C_2}{\sqrt{C_1^2 + C_2^2}}.$$

It is possible, because $x^2 + y^2 = 1$, so the point $(x, y)$ lies on the unit circle. Now, let $\varphi$ be the angle from the x-axis to the point $(x, y)$; then, by definition of cos and sin, $x = \sin \varphi$ and $y = \cos \varphi$. [See
Fig. 1] The number \( A = \sqrt{C_1^2 + C_2^2} \) is called the **amplitude**, and \( \omega t + \varphi \) is called the **phase**. The angle \( \varphi \) is called the **initial phase**.

\[
u(t) = A (\sin \varphi \cos(\omega t) + \cos \varphi \sin(\omega t)) = A \sin(\varphi t + \omega).
\]

You can view \( \nu(t) \) as the \( y \)-coordinate of a point rotating on the circle with radius \( A \), starting from \( \varphi \), with the speed of rotation \( \omega \) (and period \( T \)). [See Fig. 2]

Now, consider the case \( \gamma > 0 \). Then

\[
m u'' + \gamma u' + ku = 0.
\]

The characteristic equation is

\[
m \lambda^2 + \gamma \lambda + k = 0 \Rightarrow \lambda_{1,2} = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m}.
\]

**Case 1. Overdamping.** [Fig. 3] \( \gamma^2 - 4mk > 0, \gamma > 2\sqrt{mk} \). Then the roots are real and distinct, and the solutions are

\[
C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}.
\]

Since \( \sqrt{\gamma^2 - 4mk} < \sqrt{\gamma^2} = \gamma \), then we have:

\[
-\gamma + \sqrt{\gamma^2 - 4mk} < 0.
\]

Also,

\[
-\gamma - \sqrt{\gamma^2 - 4mk} < 0.
\]

Therefore, the roots \( \lambda_{1,2} < 0 \), and

\[
u(t) \to 0, \quad t \to \infty.
\]

In this case, damping is so strong that it does not allow oscillations to proceed, and the body goes straight to the equilibrium.

**Case 2. Underdamping.** [Fig. 4] \( \gamma^2 - 4mk < 0, \gamma < 2\sqrt{mk} \). Then the roots are imaginary:

\[
\lambda = -\frac{\gamma}{2m} \pm i\omega, \quad \omega = \frac{\sqrt{4mk - \gamma^2}}{2m}.
\]

Therefore,

\[
u(t) = C_1 e^{-(\gamma/2m)t} \cos(\omega t) + C_2 e^{-(\gamma/2m)t} \sin(\omega t).
\]

These are oscillations: damping is insufficiently strong to prevent oscillations. But they have decreasing amplitude:

\[
\sqrt{C_1^2 + C_2^2 e^{-(\gamma/2m)t}}.
\]

So this solution also tends to 0 as \( t \to \infty \). But while it tends to zero, it oscillates with frequency \( \omega \). We can write this as

\[
C_1 e^{-(\gamma/2m)t} \cos(\omega t) + C_2 e^{-(\gamma/2m)t} \sin(\omega t) = \sqrt{C_1^2 + C_2^2} e^{-(\gamma/2m)t} \sin(\omega t + \varphi),
\]

as in the previous part of the lecture.

**Case 3. Critical damping.** [Fig. 3] \( \gamma^2 - 4mk = 0, \gamma = 2\sqrt{mk} \). Then the roots are repeated: \( \lambda = -\gamma/(2m) \), and the solutions are

\[
C_1 t e^{\lambda t} + C_2 e^{\lambda t}.
\]

Since \( \lambda < 0 \), this solutions also tends to zero, just like in the case of overdamping. There are no oscillations.
\[ x = \frac{C_1}{\sqrt{C_1^2 + C_2^2}} \quad \text{and} \quad y = \frac{C_2}{\sqrt{C_1^2 + C_2^2}} \]

Fig. 1.

\[ u(t) = A \sin(\omega t + \phi) \]

Fig. 2.

Fig. 3. [Overdamping]

Fig. 4. [Underdamping]