
Consider the function
\[ u_0(t) = \begin{cases} 1, & t \geq 0; \\ 0, & t < 0; \end{cases} \]
This function is discontinuous at \( t = 0 \). Therefore, it does not have a derivative at this point. But in practice, it is still useful to consider a function \( \delta = u'_0 \).

It is equal to zero everywhere, except \( t = 0 \). At \( t = 0 \), it is equal to \( \infty \). Of course, this function does not exist. But there is a way in higher-level math to make sense of this. For now, let us just play with it.

\[
\int_{-\infty}^{+\infty} \delta(t)dt = \int_{-\infty}^{+\infty} u'_0(t)dt = u_0(t)|_{t=-\infty}^{t=\infty} = u_0(+\infty) - u_0(-\infty) = 1 - 0 = 1.
\]

In addition, for every function \( f : \mathbb{R} \rightarrow \mathbb{R} \) with \( f(+\infty) = f(-\infty) = 0 \) we have:

\[
\int_{-\infty}^{+\infty} \delta(t)f(t)dt = \int_{-\infty}^{+\infty} u'_0(t)f(t)dt = u_0(t)f(t)|_{t=-\infty}^{t=\infty} - \int_{-\infty}^{+\infty} u_0(t)f'(t)dt
\]

\[
= -\int_0^{+\infty} f'(t)dt = - f|_{t=0}^{t=\infty} = f(0).
\]

The last is the main property of \( \delta \) function:

\[
\int_{-\infty}^{+\infty} \delta(t)f(t)dt = f(0)
\]

We can also define the shifted delta function: \( \delta_c \). This is a function which is plus infinity at \( t = c \), zero everywhere else, and

\[
\int_{-\infty}^{+\infty} \delta_c(t)f(t)dt = f(c)
\]

for every function \( f : \mathbb{R} \rightarrow \mathbb{R} \) such that \( f(\pm \infty) = 0 \). This is a derivative of

\[ u_c(t) = \begin{cases} 1, & t \geq c; \\ 0, & t < c. \end{cases} \]

\[ \delta_c = u'_c(t). \]

We can also define its Laplace transform: for \( c > 0 \),

\[ L[\delta_c](s) = sL[u_c](s) - u_c(0) = se^{-cs} - 0 = e^{-cs}. \]

Sometimes we write \( \delta_c(t) = \delta(t-c) \), to indicate that this is a shifte delta function.

Solve the equation:

\[ u'' + u = \delta_3, \quad u(0) = 0, \quad u'(0) = 0. \]

Apply Laplace transform to both parts:

\[ s^2L[u](s) - su(0) - u'(0) + L[u](s) = e^{-3s}. \]

Using the initial conditions, we get:

\[ (s^2 + 1)L[u](s) = e^{-3s} \Rightarrow L[u] = \frac{e^{-3s}}{s^2 + 1}. \]

The inverse Laplace transform of this function is equal to \( u_3(t)\cos(t-3) \). So

\[ u = u_3(t)\cos(t-3). \]

Note that even though the right-hand side is ”not a function”, the left-hand side is a real function!