
An initial value problem for a differential equation is when we are given $y(t_0) = y_0$. Say,

$$y' = y, \quad y(1) = 2.$$

Then the general solution of the differential equation is $y = Ce^t$. But we need to find $C$ such that $y(1) = 2$. This $C = 2/e$. So $y = (2/e)e^t$ is the solution to this initial value problem. There are infinitely many solutions to a differential equation, so the general solution has an unknown parameter. But an initial value problem usually (but not always) has a unique solution.

Try another example: $y' = y^2$. We can consider this as the dynamics of the quantity of bacteria. If $y(t)$ is the quantity of bacteria at time $t$, and they multiply in pairs: each pair of bacterias produces another bacteria, then the rate $y'(t)$ of growth of the population is proportional to the quantity of pairs, which is $y^2(t)$.

Let us try separation of variables: if $y \neq 0$, then

$$\frac{dy}{dt} = y^2 \Rightarrow \frac{dy}{y^2} = dt \Rightarrow \int \frac{dy}{y^2} = \int dt \Rightarrow -\frac{1}{y} = t + C.$$

Solve for $y$:

$$y = -\frac{1}{t + C}, \quad \text{where } C \text{ is any real constant.}$$

But we forgot the case $y = 0$. And this is also a solution, because if $y \equiv 0$ ($y(t) = 0$ for all $t$), then $y' = 0$ (the derivative of a constant function is zero). So the general solution of the equation $y' = y^2$ is

$$y = -\frac{1}{t + C}, \quad C \text{ is any real constant, and } y = 0.$$

Solve the initial value problem $y(0) = 0$. The case $y = -1/(t + C)$ does not fit into this, because this fraction can never be zero. But the case $y = 0$ fits. So the solution to the initial value problem is $y = 0$.

Now, solve the initial value problem $y(0) = 1$. The case $y = -1/(t + C)$: plug in $t = 0$. Then $-1/C = 1$, and so

$$C = -1, \quad y = -\frac{1}{t - 1} = \frac{1}{1 - t}.$$

And the case $y = 0$ does not fit, because $y(0) = 0 \neq 1$.

**Review of Linear and Separable Equations.** A linear equation is $y' = a(t)y + b(t)$. The right-hand side should be a linear function of $y$, when $t$ is fixed. It is solved by *variation of parameters*: solve the corresponding *homogeneous equation*

$$y' = a(t)y \Rightarrow y = Ce^{\int a(t)dt},$$

and then let $C = C(t)$ and plug into the original equation. For $b \neq 0$, this equation is called *nonhomogeneous*. This distinction: homogeneous vs nonhomogeneous, is applicable only to linear equations, because it is used for variation of parameters. For nonlinear equations, it is meaningless.

A separable equation is $y' = f(t)g(y)$. We solve it by *separation of variables*:

$$\frac{dy}{dt} = f(t)g(y) \Rightarrow \frac{dy}{g(y)} = f(t)dt \Rightarrow \int \frac{dy}{g(y)} = \int f(t)dt.$$

In particular, if we have just $y' = g(y)$, then $f = 1$ and this is separable.
Let us give some examples. We answer three questions: linear? separable? homogeneous?

1. \( y' = y^2 \): nonlinear, separable, N/A.
2. \( y' = ty + t \): linear, separable, nonhomogeneous.
3. \( ty' = \sin y \): nonlinear, separable, N/A.
4. \( y' = ty \): linear, separable, homogeneous.
5. \( y' = y^2 + t \): nonlinear, non-separable, N/A.
6. \( y' = y + t \): linear, non-separable, nonhomogeneous.
7. \( y' = e^{y+t} \): nonlinear, separable, N/A.
8. \( y' = ty - 1 \): linear, non-separable, nonhomogeneous.