Problem 1. Consider mechanical vibrations with mass $m = 2$, damping $\gamma = 1$, and $k = 2$. If there are no external forces, find the general solution of the equation.

Solution. We have: $mu'' + \gamma u' + ku = 0$, so $2u'' + u' + 2u = 0$. Characteristic equation: $2\lambda^2 + \lambda + 2 = 0$, and so 

$$
\lambda = \frac{-1 \pm \sqrt{1 - 4 \cdot 2 \cdot 2}}{2 \cdot 2} = \frac{-1 \pm \sqrt{-15}}{4} = \frac{-1 \pm \sqrt{15}}{4}i.
$$

So the general solution is 

$$
 u(t) = C_1 e^{-t/4} \cos \left( \frac{\sqrt{15}}{4} t \right) + C_2 e^{-t/4} \sin \left( \frac{\sqrt{15}}{4} t \right)
$$

Problem 2. Consider an electrical circuit with

$C = 1/2$, $R = 3$, $L = 1$, $E_0(t) = 2 \cos(t) + \sin(t)$.

Find the steady-state solution. (Hint: you do not need to find a general solution for this!)

Solution. We have:

$$
Q'' + 3Q' + 2Q = E_0(t) = 2 \cos t + \sin t.
$$

Therefore, the steady-state solution can be found in the form 

$$
Q(t) = A \cos t + B \sin t.
$$

Plugging in the equation, we can find that 

$$
Q' = -A \sin t + B \cos t, \quad Q'' = -A \cos t - B \sin t.
$$

Therefore, 

$$
Q'' + 3Q' + 2Q = (A + 3B) \cos t + (B - 3A) \sin t.
$$

Comparing it with the right-hand side, we have:

$$
A + 3B = 2, \quad B - 3A = 1.
$$

Solving this equation, we get: 

$$
A = -\frac{1}{10}, \quad B = \frac{7}{10}.
$$

Therefore, the steady-state solution is 

$$
 u(t) = -\frac{1}{10} \cos t + \frac{7}{10} \sin t
$$