Example 1. $y'' - 2y' + 2y = te^{2t}$. Recall the basic principle: general solution $y_0$ of the corresponding homogeneous equation

$$y'' - 2y' + 2y = 0$$

plus a particular solution $y_1$ of the nonhomogeneous equation equals general solution of nonhomogeneous equation.

First, find $y_0$. Characteristic equation:

$$\lambda^2 - 2\lambda + 2 = 0 \quad \Rightarrow \quad \lambda = 1 \pm i.$$ 

Therefore, the general solution is

$$y_0(t) = C_1 e^{(1+i)t} + C_2 e^{(1-i)t}.$$ 

We can rewrite it using Euler’s formula:

$$e^{x+iy} = e^x(\cos y + i \sin y).$$

We have:

$$y_0(t) = C_1 e^t(\cos t + i \sin t) + C_2 e^t(\cos t - i \sin t) = (C_1 + C_2)e^t \cos t + i(C_1 - C_2)e^t \sin t = K_1 e^t \cos t + K_2 e^t \sin t.$$ 

Since $C_1$ and $C_2$ are arbitrary constants, $K_1$ and $K_2$ are also arbitrary constants. You can write just this last form, no need to reduce it as above.

Now, let us find a particular solution. Since 2 in the exponent $te^{2t}$ does not coincide with $\lambda = 1 \pm i$, we need to take

$$y_1(t) = (At + B)e^{2t},$$

where $A$ and $B$ are unknown coefficients. Plug this into the equation and compare coefficients. You will get a system of equations for $A$ and $B$.

Example 2. $y'' - 2y' + 2y = \cos t + 2 \sin t$. The right-hand side is actually the sum of complex exponents:

$$e^{it} = \cos t + i \sin t, \quad e^{-it} = \cos t - i \sin t.$$ 

Therefore,

$$\cos t = \frac{1}{2} (e^{it} + e^{-it}), \quad \sin t = \frac{1}{2i} (e^{it} - e^{-it}).$$

Therefore, we need to seek solution

$$y_1 = Ae^{it} + Be^{-it} = A(\cos t + i \sin t) + B(\cos t - i \sin t) = (A + B) \cos t + i(A - B) \sin t = A_1 \cos t + B_1 \sin t,$$

where $A, B$ and $A_1 = A + B, A_1 = i(A - B)$ are unknown coefficients. You can just write

$$y_1 = A_1 \cos t + B_1 \sin t$$

without doing the explanation above. Plug this into the equation:

$$y_1' = -A_1 \sin t + B_1 \cos t, \quad y_1'' = -A_1 \cos t - B_1 \sin t.$$ 

$$y_1'' - 2y_1' + 2y_1 = (-A_1 + 2B_1 + 2A_1) \cos t + (-B_1 - 2A_1 + 2B_1) \sin t.$$ 

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\[ = (A_1 + 2B_1) \cos t + (B_1 - 2A_1) \sin t. \]

This must be equal to \( \cos t + 2 \sin t \). Compare coefficients:

\[
A_1 + 2B_1 = 1, \quad B_1 - 2A_1 = 2.
\]

Solve for \( A_1 \) and \( B_1 \) and finish the proof.

**Example 3.** \( y'' - 2y' + 2y = e^t \sin t \). The right-hand side is the combination of two exponents \( e^{(1 \pm i)t} \), and \( 1 \pm i \) are exactly the roots of the characteristic equation. So we need to raise the degree of polynomial, from zero to one:

\[ y_1 = (At + B)e^t \cos t + (Ct + D)e^t \sin t. \]

We need to put both \( \cos \) and \( \sin \) in \( y_1 \), because even though we get only one function \( \sin \) in the right-hand side, it contains both exponents, so there should be two exponents (or, equivalently, two trig functions) in \( y_1 \).

**Example 4.** \( y'' - 2y' + 2y = e^{2t} \cos t + te^t \sin t \). The part \( e^{2t} \cos t \) corresponds to exponents with \( 2 \pm i \), and these numbers are different from \( 1 \pm i \) (the roots of the characteristic equation). So we do not need to raise the degree of the polynomial (now it is zero). This part of the right-hand side gives us

\[ y_1 = A_1 e^{2t} \cos t + B_1 e^{2t} \sin t. \]

The part \( te^t \sin t \) corresponds to the exponents with \( 1 \pm i \), which coincide with roots of the characteristic equation. So we need to raise the degree of polynomial, from one to two:

\[ y_1 = (At^2 + Bt + C)e^t \cos t + (Dt^2 + Et + F)e^t \sin t. \]

The final expression for \( y_1 \) is

\[ y_1 = A_1 e^{2t} \cos t + B_1 e^{2t} \sin t + (At^2 + Bt + C)e^t \cos t + (Dt^2 + Et + F)e^t \sin t. \]