Consider the following model. A body of mass \( m \) is moving horizontally on a spring. It slides on a smooth horizontal surface. The position of the body is \( u \) (where 0 is the equilibrium). Two forces act upon this body:

- the spring force, which wants to return this body to equilibrium: \( F_1 = -ku \), proportional to the distance from the equilibrium. \( k > 0 \) is a coefficient. The minus sign stands because the force wants to pull the body back to the equilibrium;
- damping: air or water resistance, proportional to the speed: \( F_2 = -\gamma u' \), where \( \gamma > 0 \). The minus sign indicates that the force wants to slow it down;
- external force \( F_0(t) \).

By Newton’s Second Law, mass times acceleration equals the sum of all forces acting upon this body:

\[
mu'' = F_1 + F_2 = -ku - \gamma u' + F_0(t).
\]

Therefore,

\[
mu'' + \gamma u' + ku = F_0(t).
\]

Assume \( F_0(t) = G_1 \cos(\omega_0 t) + G_2 \sin(\omega_0 t) \). Let us solve this equation.

Assume there is no damping: \( \gamma = 0 \). Then we have:

\[
mu'' + ku = G_1 \cos(\omega_0 t) + G_2 \sin(\omega_0 t).
\]

Solving the homogeneous equation as in Lecture 17, we have:

\[
u = C_1 \cos(\omega t) + C_2 \sin(\omega t), \quad \omega = \sqrt{\frac{k}{m}}.
\]

This has the following physical interpretation: In the absence of damping, and without external force, the system oscillates with frequency \( \omega = \sqrt{k/m} \).

If \( \omega_0 \neq \omega \), then we can try

\[
A \cos(\omega_0 t) + B \sin(\omega t)
\]
as a particular solution of the nonhomogeneous equation, and find \( A \) and \( B \) by plugging in the equation. Then the general solution of the nonhomogeneous equation is

\[
u(t) = A \cos(\omega_0 t) + B \sin(\omega t) + C_1 \cos(\omega t) + C_2 \sin(\omega t).
\]

If \( \omega = \omega \), then this does not work, because plugging in

\[
A \cos(\omega_0 t) + B \sin(\omega t)
\]
gives you zero in the right-hand side: this is already a solution to the homogeneous equation. Now, let us raise the degree of polynomial by 1:

\[
A t \cos(\omega_0 t) + B t \sin(\omega t).
\]

Then you can find the numerical values of \( A \) and \( B \). Note that these oscillations have amplitude

\[
\sqrt{A^2 + B^2 t},
\]

which is increasing up to infinity.

This has the following physical meaning.
When the internal frequency (=the frequency of oscillations in absence of outside forces) coincides with the frequency of outside force, then the oscillations have increasing amplitude which goes to infinity.

This phenomenon is called a resonance. This explains the well-known stories about soldiers marching on a bridge and destroying this bridge. The frequency of their footsteps was the same as the frequency of oscillations of the bridge by itself. (A bridge is moving a very little bit, even in the absence of outside pressure, but we usually do not notice this.) Sometimes the resonance is an undesirable phenomenon, as in this case; in other cases, it is helpful.

Now, consider the case of damping. The general theory is in Section 3.8 in the book. We shall not develop it here. Rather, we shall consider a concrete example. Let \( m = 1 \), \( \gamma = 2 \), \( k = 3 \), \( F(t) = 2 \cos t \). Then
\[
    u'' + 2u' + 3u = 2 \cos t.
\]

First, let us solve the homogeneous equation
\[
    u'' + 2u' + 3u = 0.
\]

The characteristic equation is \( \lambda^2 + 2\lambda + 3 = 0 \). Its roots are \( \lambda_{1,2} = -1 \pm \sqrt{2}i \). The general solution is
\[
    C_1 e^{-t} \cos(\sqrt{2}t) + C_2 e^{-t} \sin(\sqrt{2}t).
\]

Now, let us find a particular solution of the nonhomogeneous equation. The right-hand side \( 2 \cos t \) corresponds to \( e^{\pm it} \). Since \( -1 \pm \sqrt{2}i \neq \pm i \), we do not need to raise the degree of the polynomial. It was of degree zero (the constant 2), and it will be of degree zero, i.e. constant. Let us look for
\[
    u = A \cos t + B \sin t.
\]

After plugging into the equation and comparing coefficients, we get: \( A = B = 1/2 \). Therefore, the answer is
\[
    u(t) = \frac{1}{2} \cos t + \frac{1}{2} \sin t + C_1 e^{-t} \cos(\sqrt{2}t) + C_2 e^{-t} \sin(\sqrt{2}t).
\]

It consists of the term \( C_1 e^{-t} \cos(\sqrt{2}t) + C_2 e^{-t} \sin(\sqrt{2}t) \), which goes to zero as \( t \to \infty \), and the term \( \frac{1}{2} \cos t + \frac{1}{2} \sin t \), which is an oscillation and which does not go to zero. This last term is called steady-state solution. Because we can set \( C_1 = C_2 = 0 \), and we will have
\[
    u(t) = \frac{2}{5} \cos t + \frac{4}{5} \sin t.
\]

Always, when \( t \to \infty \), \( u(t) \) is very close to \( \frac{1}{2} \cos t + \frac{1}{2} \sin t \), because the other part tends to zero.