Practice Problems for the Final Exam
Math 307J. Autumn 2014

In Problems 1-6, classify the equation (linear/nonlinear, separable, autonomous, if linear: homogeneous/nonhomogeneous). Find the general solution of the equation. If an initial condition is given, solve the initial value problem.

**Problem 1.** \(ty' = 4y + 3\).

**Problem 2.** \(y' = y^2 + 2\).

**Problem 3.** \(y' = 1 - y^2, \ y(0) = -1\).

**Problem 4.** \(y' = e^{t-y}, \ y(1) = 1\).

**Problem 5.** \(y' = -2ty^4\).

**Problem 6.** \(y' = 3ty + t^3\).

**Problem 7.** Solve the equation \(y' = -2ty\) with the initial condition \(y(0) = 1\) and then use Euler’s method with step: (i) \(h = 1\); (ii) \(h = 0.5\) to find approximately \(y(1)\). Find the relative error of the approximation.

**Problem 8.** Analyze asymptotically the following autonomous equation:
\[
y' = -3(y - 1)^2(y - 2)^2y.
\]

**Problem 9.** Consider a pool of volume 100, initially filled with petroleum. Clean water flows in this pool with speed 1 per minute, and the mixture flows out of this pool with the same speed.
(i) Find the differential equation for the amount \(y(t)\) of waste in the pool at time \(t\).
(ii) Analyze asymptotically the equation and find the limit \(y(t)\) as \(t \to \infty\).
(iii) Solve the initial value problem.

**Problem 10.** A student loan is 50000. You pay 500 every month, continuously. Annual interest is 6%, accrued continuously. However, the first two years are called the grace period: the interest does not accrue. When will you repay the loan?

**Problem 11.** A ball of mass \(m\) moves in water, without gravity. Its initial speed is \(v_0\). The resistance force is \(kv\). When will the ball stop? How long will the ball travel until it stops?

In Problems 12-15, find the general solution of the second-order differential equation.

**Problem 12.** \(y'' + y' - 6y = e^t\);

**Problem 13.** \(2y'' - 2y = 3e^t + \cos t\);

**Problem 14.** \(y'' + 2y' + y = t - e^t\);
Problem 15. \( y'' + 2y' + 5y = \sin(2t) \).

In Problems 16-17, find the general solution of the second-order differential equation, but do not find the precise values of undetermined coefficients. Denote them by letters: \( A_1, A_2, A_3, \ldots \). Denote constants by \( C_1, C_2 \).

Problem 16. \( y'' + 2y' + 10y = 4t^2e^{-t}\cos(3t) + t^5 - t + t^3e^{2t} \).

Problem 17. \( y'' - 6y' + 9y = -4t^2e^{3t} + t^2e^t\sin(3t) \).

Problem 18. Using the method of variation of parameters, find the general solution of
\[ t^2y'' + 5ty' + 4y = t, \]
if one of the solutions of the corresponding homogeneous equation is \( y_1(t) = t^{-2} \).

Problem 19. Consider a mechanical system with \( m = 2, \ k = 4, \ \gamma = 4 \). Let the external force be \( F_0(t) = 2\sin(2t) \). Find the steady-state solution.

Problem 20. Consider an electrical system with \( C = 3, \ R = 0, \ L = 2 \). For which \( \omega \) is there a resonance? Assuming \( E(t) = \cos(\omega t), \ Q(0) = I(0) = 0 \), find \( Q(t) \) for all \( t \).

In Problems 21-24, find the Laplace transforms of the function \( f \).

Problem 21. \( f = e^t\delta_2(t) \).

Problem 22. \( f = \cos(t)e^{3t} - t \).

Problem 23. \( f = u_3(t)e^{t-3} \).

Problem 24. \( f = e^{2t}\sin(3t)u_1(t) \).

In Problems 25-27, find the inverse Laplace transforms of the function \( F \).

Problem 25. \( F = \frac{s+1}{s^2+2s+2} \).

Problem 26. \( F = \frac{s^2}{s^2-2s+5}e^{-3s} \).

Problem 27. \( F = \frac{s^2}{s(1-s)}e^{-s} \).

In Problems 28-30, solve the equation using the Laplace transforms.

Problem 28. \( y'' - 2y' + y = e^t, \ y(0) = y'(0) = 0 \).

Problem 29. \( y'' - y = e^{2t}u_3(t), \ y(0) = 1, \ y'(0) = 0 \).

Problem 30. \( y'' + 4y = e^t\delta_3(t) \).