Problem 1. Find the general solution of the equation

\[ y'' + 4y' + 4y = (2t^2 + 3)e^{-2t} + (t + 2)e^{-2t} \cos(3t) + 2 \sin(4t) - 4e^t - 3t. \]

Do not find undetermined coefficients, just designate them by letters \((A, B, \ldots)\).

Solution. First, solve the homogeneous equation

\[ y'' + 4y' + 4y = 0. \]

Characteristic equation: \(\lambda^2 + 4\lambda + 4 = 0\), double root \(\lambda = -2\). General solution:

\[ y_0 = C_1 t e^{-2t} + C_2 e^{-2t}. \]

Now, let us solve the nonhomogeneous equation. The term \((2t^2 + 3)e^{-2t}\) corresponds to the exponent \(-2\), which coincides with the double root. So we need to raise the degree 2 of polynomial by 2. The resulting polynomial will have degree 4:

\[ y_1 = (At^4 + Bt^3 + Ct^2)e^{-2t}. \]

We could add \(Dt + E\), but anyway they are already included into the general solution of the nonhomogeneous equation. The next term corresponds to the exponents \(e^{(-2\pm3i)t}\). But \(-2 \pm 3i \neq -2\), so we do not need to raise the degree 1 of the polynomial \(t + 2\). However, we need to include both \(\cos\) and \(\sin\), because we have two exponents here: \(e^{(-2\pm3i)t}\). We have:

\[ y_2 = (A_1 t + B_1)e^{-2t} \cos(3t) + (A_2 t + B_2)e^{-2t} \sin(3t). \]

Similarly, \(2 \sin(4t)\) corresponds to \(e^{\pm4it}\). And \(\pm4i \neq -2\). Therefore, we do not need to raise the degree of the polynomial, which in this case is 0:

\[ y_3 = D_1 \cos(4t) + D_2 \sin(4t). \]

Moreover, \(4e^t\) corresponds to the exponent 1, which is not equal to 2. Therefore, we do not need to raise the degree of the polynomial, which is zero. We have:

\[ y_4 = E e^t. \]

Finally, \(-3t\) corresponds to the exponent \(e^{0t}\). But \(0 \neq -2\), so we do not need to raise the degree of polynomial. We have:

\[ y_5 = F t + G. \]

The general solution of the nonhomogeneous equation is

\[ y = y_0 + y_1 + y_2 + y_3 + y_4 + y_5. \]