1. Linear Systems I. April 1, 2013

1.1. Reduction Example

Consider a system

\[
\begin{align*}
    x_1 - x_2 + x_4 &= 1 \\
    x_2 + 2x_3 - x_4 &= 0 \\
    2x_1 + x_2 + x_3 + x_4 &= 2 \\
\end{align*}
\]

Let us write its coefficients in the form of a $3 \times 4$-matrix:

\[
A = \begin{bmatrix}
    1 & -1 & 0 & 1 \\
    0 & 1 & 2 & -1 \\
    2 & 1 & 1 & 1 \\
\end{bmatrix}, \quad \text{and} \quad b = \begin{bmatrix}
    1 \\
    0 \\
    2 \\
\end{bmatrix}
\]

Consider an augmented matrix:

\[
[A \mid b] = \begin{bmatrix}
    1 & -1 & 0 & 1 & 1 \\
    0 & 1 & 2 & -1 & 0 \\
    2 & 1 & 1 & 1 & 2 \\
\end{bmatrix}
\]

Denote the equations by $E_1, E_2, E_3$. Let us eliminate the first element in the third row. Multiply the first equation by $-2$ and add it to the third equation: $E_3 := E_3 + (-2)E_1$. Then we have:

\[
\begin{bmatrix}
    1 & -1 & 0 & 1 & 1 \\
    0 & 1 & 2 & -1 & 0 \\
    0 & 3 & 1 & -1 & 0 \\
\end{bmatrix}
\]

Let us eliminate the second element in the third row. Multiply the second equation $-3$ and add it to the third equation: $E_3 := E_3 + (-3)E_2$. Then we have:

\[
\begin{bmatrix}
    1 & -1 & 0 & 1 & 1 \\
    0 & 1 & 2 & -1 & 0 \\
    0 & 0 & -5 & 2 & 0 \\
\end{bmatrix}
\]

Multiply the third equation by $-1/5$: $E_3 := (-1/5)E_3$. Then we have:

\[
\begin{bmatrix}
    1 & -1 & 0 & 1 & 1 \\
    0 & 1 & 2 & -1 & 0 \\
    0 & 0 & 1 & -2/5 & 0 \\
\end{bmatrix}
\]

Now the matrix is in echelon form. Precise definitions are given below. Let us reduce it further to reduced echelon form. Add the third equation times $-2$ to the second one: $E_2 := E_2 + (-2)E_3$.

\[
\begin{bmatrix}
    1 & -1 & 0 & 1 & 1 \\
    0 & 1 & 0 & -1/5 & 0 \\
    0 & 0 & 1 & -2/5 & 0 \\
\end{bmatrix}
\]

Add the second equation to the first one: $E_1 := E_1 + E_2$.

\[
\begin{bmatrix}
    1 & 0 & 0 & 4/5 & 1 \\
    0 & 1 & 0 & -1/5 & 0 \\
    0 & 0 & 1 & -2/5 & 0 \\
\end{bmatrix}
\]

This is a reduced echelon form.
1.2. Finalizing the Solution

Let us solve it:

\[
\begin{align*}
  x_1 + \frac{4}{5}x_4 &= 1 \\
  x_2 - \frac{1}{5}x_4 &= 0 \\
  x_3 - \frac{2}{5}x_4 &= 0.
\end{align*}
\]

The variable \(x_4\) is free (unconstrained), while other variables are constrained. This free variable can take any real values: let \(x_4 = a \in \mathbb{R}\). Then

\[x_1 = 1 - \frac{4}{5}a, \quad x_2 = \frac{1}{5}a, \quad x_3 = \frac{2}{5}a.\]

This system has infinitely many solutions:

\[x_1 = 1 - \frac{4}{5}a, \quad x_2 = \frac{1}{5}a, \quad x_3 = \frac{2}{5}a, \quad x_4 = a, \quad a \in \mathbb{R}.\]

For example, let \(a = 0\): \((1, 0, 0, 0)\); let \(a = 1\): \((-3, 1, 2, 5)\).

1.3. Definitions

A matrix is in **echelon form** if:

1. All rows that consist entirely of zeros are grouped together at the bottom of the matrix.
2. In every nonzero row, the first nonzero entry (counting from left to right) is a 1.
3. If the \((i+1)\)st row contains nonzero entries, then the first nonzero entry is in a column to the right of the first nonzero entry in the \(i\)th row.

A matrix is in **reduced echelon form** if, in addition to echelon form, every unit which is the first nonzero entry in any row is the only nonzero element in its column.

Three **row elementary operations**:

1. interchanging the equations (=rows);
2. multiplying one equation by a nonzero scalar;
3. adding one equation times a scalar to another.

They lead to equivalent systems of equations.