11. Basis and Dimension. April 24, 2013

11.1. Example

Consider three vectors

\[ v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 3 \\ 4 \\ 5 \\ 6 \end{bmatrix} \]

What is the dimension of the linear subspace?

\[ L = \operatorname{Sp}\{v_1, v_2, v_3\} \]

It seems that the answer is three, because it is spanned by three vectors. However, the correct answer is two, because these vectors are linearly dependent:

\[ v_2 = \frac{1}{2}(v_1 + v_3). \]

So every linear combination of \( v_1, v_2, v_3 \) can be written as a linear combination of \( v_1, v_3 \) only:

\[ x_1v_1 + x_2v_2 + x_3v_3 = x_1v_1 + x_2\frac{1}{2}(v_1 + v_3) + x_3v_3 = (x_1 + \frac{1}{2}x_2)v_1 + (x_3 + \frac{1}{2}x_2)v_3. \]

To find the dimension, you should extract a minimal spanning system of vectors, in this case \( v_1, v_3 \). Such system is called a basis, and the quantity of vectors is called the dimension. A basis is a system of vectors which spans the subspace, and is linearly independent. Minimality means precisely that it is linearly independent.

Note that \( v_1, v_2 \) is also a basis, because \( v_3 = 2v_2 - v_1 \) is expressible as a linear combination of \( v_1, v_2 \). Similarly, \( v_2, v_3 \) can also serve as a basis.

Do not confuse the dimension of a subspace with the dimension of the general space (i.e. the number of components in each vector). Here, the dimension of the embedding space is four, but the dimension of this subspace \( L \) is two.

11.2. General Procedure of Finding a Basis

Let us find which vectors \( v_1, v_2, v_3 \) are expressible as linear combinations of others. Consider the system of equations

\[ x_1v_1 + x_2v_2 + x_3v_3 = 0. \]

Write it in matrix form and solve:

\[
\begin{bmatrix}
1 & 2 & 3 & 0 \\
2 & 3 & 4 & 0 \\
3 & 4 & 5 & 0 \\
4 & 5 & 6 & 0
\end{bmatrix} \Rightarrow \begin{bmatrix}
1 & 2 & 3 & 0 \\
0 & -1 & -2 & 0 \\
0 & -2 & -4 & 0 \\
0 & -3 & -6 & 0
\end{bmatrix} \Rightarrow \begin{bmatrix}
1 & 2 & 3 & 0 \\
0 & 1 & 2 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} \Rightarrow \begin{bmatrix}
1 & 0 & -1 & 0 \\
0 & 1 & 2 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

Then we have:

\[ x_1 - x_3 = 0, \quad x_2 + 2x_3 = 0. \]

The variable \( x_3 \) is free, it corresponds to unnecessary vectors, which should be thrown out of the basis. The variables \( x_1 \) and \( x_2 \) are constrained, they correspond to essential vectors, which should
remain in the basis. Unnecessary vectors can be expressed as a linear combination of necessary ones. Indeed, let $x_3 = 1$. Then $x_1 = 1$ and $x_2 = -2$. So we have:

$$x_1v_1 + x_2v_2 + x_3v_3 = 0 \Rightarrow v_3 = 2v_2 - v_1.$$ 

Note: this method only gives us that $v_3$ can be removed, and $v_1$ and $v_2$ should remain in the basis. We know that for this particular situation, we can also throw away $v_1$, or $v_2$, and the remaining two vectors also form a basis. However, our method does not guarantee this in general case.

**Bases of $\mathbb{R}^n$**

The whole space $\mathbb{R}^n$ has the *standard basis* $e_1, \ldots, e_n$. Let us describe it in case $n = 4$:

$$
e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \ e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \ e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \ e_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

In general, for any invertible $n \times n$-matrix $A$ its columns $v_1, \ldots, v_n$ form a basis for $\mathbb{R}^n$, because they are linearly independent, and any vector $b$ is representable as a linear combination $x_1v_1 + \ldots + x_nv_n = Ax$. Indeed, just let $x = A^{-1}b$. 
