
14.1. Definition

A basis is called *orthogonal* if any two of its vectors are perpendicular. A basis is called *orthonormal* if, in addition, each vector has length one (so it is a unit vector).

For example, \( v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \) and \( v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \) form an orthogonal basis of \( \mathbb{R}^2 \), and \( w_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \) and \( w_2 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} \) constitute an orthonormal basis of \( \mathbb{R}^2 \).

14.2. Coordinates

Suppose we want to find the coordinates of the vector \( b = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \) in this basis. In other words, we wish to find \( x_1, x_2 \) such that

\[ b = x_1v_1 + x_2v_2. \]

We can solve the system of equations

\[
\begin{bmatrix}
1 & 1 \\
1 & -1
\end{bmatrix}
\begin{bmatrix}
x_1 \\ x_2
\end{bmatrix} =
\begin{bmatrix}
2 \\ 1
\end{bmatrix}
\]

in the usual way, by reducing it to REF. But here we can use the fact that this basis is orthogonal, so \( v_1 \cdot v_2 = 0 \). Multiply the equation \( b = x_1v_1 + x_2v_2 \) by \( v_1 \):

\[ b \cdot v_1 = x_1v_1 \cdot v_1 + x_2v_2 \cdot v_1 = x_1v_1 \cdot v_1, \Rightarrow x_1 = \frac{b \cdot v_1}{v_1 \cdot v_1} = \frac{3}{2}. \]

The same can be done with \( x_2 \):

\[ b \cdot v_2 = x_1v_1 \cdot v_2 + x_2v_2 \cdot v_2 = x_2v_2 \cdot v_2 \Rightarrow x_2 = \frac{b \cdot v_2}{v_2 \cdot v_2} = \frac{1}{2}. \]

Here, we are doing orthogonal projections of the vector \( b \) onto the vectors \( v_1, v_2 \). It is like orthogonal projections of \( b \) onto the coordinate axes, because orthogonal vectors serve as kind of new coordinate axes.

14.3. Making an Orthogonal Basis

Suppose we have a basis

\[ v_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \]

We would like to construct an orthogonal basis from it. Let us find it in the form:

\[ w_1 = v_1, \quad w_2 = v_2 + \alpha w_1, \quad w_3 = v_3 + \beta w_1 + \gamma w_2. \]

So we are projecting the second vector \( v_2 \) on the direction which is orthogonal to the first vector. And we are projecting the third vector \( v_3 \) on the direction orthogonal to the first two vectors. First, we make the first two vectors perpendicular, then we do the same with the third one.
Here, $\alpha, \beta, \gamma$ are coefficients which have to be determined. We want: $w_2 \cdot w_1 = 0$. So

$$v_1 \cdot v_2 + \alpha v_1 \cdot v_1 = 0 \Rightarrow \alpha = -\frac{v_1 \cdot v_2}{v_1 \cdot v_1} = -\frac{1}{2}.$$  

And

$$w_2 = v_2 + \alpha w_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1/2 \\ 1/2 \end{bmatrix}$$

Moreover,

$$w_1 \cdot w_3 = 0 \Rightarrow \beta w_1 \cdot w_1 + \gamma w_2 \cdot w_1 + v_3 \cdot w_1 = 0 \Rightarrow \beta w_1 \cdot w_1 + v_3 \cdot w_1 = 0 \Rightarrow \beta = -\frac{v_3 \cdot w_1}{w_1 \cdot w_1} = -\frac{1}{2},$$

and

$$w_2 \cdot w_3 = 0 \Rightarrow \beta w_1 \cdot w_2 + \gamma w_2 \cdot w_2 + v_3 \cdot w_2 = 0 \Rightarrow \gamma w_2 \cdot w_2 + v_3 \cdot w_2 = 0 \Rightarrow \gamma = -\frac{v_3 \cdot w_2}{w_2 \cdot w_2} = -\frac{1}{3}.$$  

Thus,

$$w_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{3} \begin{bmatrix} 1 \\ -1/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 2/3 \\ -2/3 \end{bmatrix}$$

So we got the following orthogonal basis:

$$w_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad w_2 = \begin{bmatrix} 1 \\ -1/2 \\ 1/2 \end{bmatrix}, \quad w_3 = \begin{bmatrix} 2/3 \\ 2/3 \\ -2/3 \end{bmatrix}$$