
15.1. Rotations

Consider the matrix

\[ A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \]

How does it act on vectors \( x \in \mathbb{R}^2 \)? We have:

\[ x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow Ax = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix} \]

For example, if \( x = [1, 0]^T \), then \( Ax = [0, 1]^T \); if \( x = [0, 1]^T \), then \( Ax = [-1, 0]^T \); if \( x = [1, 1]^T \), then \( Ax = [-1, 1]^T \). This is the rotation by \( 90^\circ = \pi/2 \) counter-clockwise.

Consider the matrix

\[ A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \]

How does it act on vectors \( x \in \mathbb{R}^2 \)? We have:

\[ x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow Ax = \begin{bmatrix} x_2 \\ -x_1 \end{bmatrix} \]

For example, if \( x = [1, 0]^T \), then \( Ax = [0, -1]^T \); if \( x = [0, 1]^T \), then \( Ax = [1, 0]^T \); if \( x = [1, 1]^T \), then \( Ax = [1, -1]^T \). This is the rotation by \( 90^\circ = \pi/2 \) clockwise, or, equivalently, by \( 270^\circ = 3\pi/2 \) counter-clockwise.

Finally, consider the more general matrix

\[ A = \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix} \]

How does it act on a vector \( x = [x_1 x_2]^T \)? Let us write it in polar coordinates:

\[ x = \begin{bmatrix} r \cos \theta \\ r \sin \theta \end{bmatrix} \]

Then

\[ Ax = \begin{bmatrix} r \cos \theta \cos \alpha - r \sin \theta \sin \alpha \\ r \cos \theta \sin \alpha + r \sin \theta \cos \alpha \end{bmatrix} = \begin{bmatrix} r \cos(\alpha + \theta) \\ r \sin(\alpha + \theta) \end{bmatrix} \]

So the length \( r = \sqrt{x_1^2 + x_2^2} \) of the vector remains unchanged, and the angle \( \theta \) between this vector and the positive \( x_1 \)-axis increases by \( \alpha \). This is the rotation by the angle \( \alpha \) counter-clockwise.

15.2. Symmetrical Reflections

Consider the matrix

\[ A = \begin{bmatrix} \cos \alpha & \sin \alpha \\ \sin \alpha & -\cos \alpha \end{bmatrix} \]

Again, write \( x \in \mathbb{R}^2 \) in the same polar coordinate form and repeat the computations:

\[ Ax = \begin{bmatrix} r \cos \theta \cos \alpha + r \sin \theta \sin \alpha \\ r \cos \theta \sin \alpha - r \sin \theta \cos \alpha \end{bmatrix} = \begin{bmatrix} r \cos(\alpha - \theta) \\ r \sin(\alpha - \theta) \end{bmatrix} \]
So the length \( r = \sqrt{x_1^2 + x_2^2} \) of the vector remains unchanged, and the angle \( \theta \) between this vector and the positive \( x_1 \)-axis becomes \( \alpha - \theta \). For example, suppose that \( \alpha = \pi/2 \). Then \( \theta = 0 \) is mapped to \( \pi/2 \), \( \pi/4 \) is mapped into itself, \( -\pi/4 \) is mapped to \( 3\pi/4 \), etc. So this is the symmetrical reflection with respect to the line \( x_1 = x_2 \), which makes the angle \( \pi/4 \) with the \( x_1 \)-axis and passes through the origin. In general, the matrix above is the symmetrical reflection with respect to the line that passes through the origin and forms the angle \( \alpha \) with the \( x_1 \)-axis.

**15.3. Examples.**

Rotation by \( \pi/4 \) counterclockwise is given by the matrix

\[
A = \begin{bmatrix} \cos(\pi/4) & -\sin(\pi/4) \\ \sin(\pi/4) & \cos(\pi/4) \end{bmatrix} = \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}
\]

Reflection with respect to the \( x_2 \)-axis: \( \alpha/2 = 90^\circ = \pi/2 \), so \( \alpha = \pi \), and

\[
A = \begin{bmatrix} \cos \pi & \sin \pi \\ \sin \pi & -\cos \pi \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}
\]

**15.4. Orthogonal Matrices**

Each of these matrices has columns which form an orthonormal basis. Because they are orthogonal to each other and each of them has length one. Such matrices are called *orthogonal*. They have a number of important properties: for example, as we have seen, they preserve the magnitude of a vector on which they act. We will study them later.