17. Least Squares II. May 15, 2013

17.1. Least Distance to Subspace

Find the smallest distance from the vector

\[ b = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \]

to the plane \( W = \{ x \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = 0 \} \). Let us find the basis for this subspace:

\[ x = \begin{bmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \]

so the basis consists of two vectors:

\[ v_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad v_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \]

Because these vectors are linearly independent, since they are not proportional, and because we can express any vector \( x \in W \) as their linear combination, we have: \( \{v_1, v_2\} \) is the basis for \( W \).

We would like to find \( x_1, x_2 \) such that the distance from \( b \) to \( x_1 v_1 + x_2 v_2 \) is smallest. This will be the case if \( b - x_1 v_1 - x_2 v_2 \) is orthogonal to the plane generated by \( v_1, v_2 \), or, equivalently, is orthogonal to both of these vectors. So we must have:

\[(b - x_1 v_1 - x_2 v_2) \cdot v_1 = 0, \quad (b - x_1 v_1 - x_2 v_2) \cdot v_2 = 0,\]

or

\[ x_1 v_1 \cdot v_1 + x_2 v_1 \cdot v_2 = v_1 \cdot b, \quad x_1 v_2 \cdot v_1 + x_2 v_2 \cdot v_2 = v_2 \cdot b, \]

so the system is

\[
\begin{bmatrix} v_1 \cdot v_1 & v_1 \cdot v_2 \\ v_2 \cdot v_1 & v_2 \cdot v_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} v_1 \cdot b \\ v_2 \cdot b \end{bmatrix}
\]

Plug in the numbers:

\[
\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}
\]

The solution is \( x_1 = 0, \ x_2 = 1 \). So the best approximation to \( b \) on the plane \( W \) is the vector \( x_1 v_1 + x_2 v_2 = v_2 \). Its distance from \( b \) is equal to

\[ |b - v_2| = \sqrt{(1 - (-1))^2 + (2 - 0)^2 + (3 - 1)^2} = \sqrt{12} = 2\sqrt{3} \]

17.2. Least Squares Exponential Fit

Assume we have:

\[
\begin{array}{c|ccc}
  t & 0 & 1 & 2 & 3 \\
  \hline
  y & 1 & 1.3 & 1.5 & 1.7
\end{array}
\]
We would like to reconstruct \( y(t) = ce^{kt} \). For example, this data might be the Consumer Price Index (inflation) for these years 0-3. We cannot directly apply least squares method here, since \( y \) does not depend on \( k \) linearly. But let us take logarithms:

\[
\log y = \log c + kt.
\]

Then we have:

<table>
<thead>
<tr>
<th>( t )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \log y )</td>
<td>0</td>
<td>1.3</td>
<td>1.5</td>
<td>1.7</td>
</tr>
</tbody>
</table>

So we have the system:

\[
\begin{align*}
\log c + k0 &= 0 \\
\log c + k1 &= \log 1.3 \\
\log c + k2 &= \log 1.5 \\
\log c + k3 &= \log 1.7
\end{align*}
\]

Then we should find \( c, k \) for the shortest distance from

\[
b = \begin{bmatrix} 0 \\ \log 1.3 \\ \log 1.5 \\ \log 1.7 \end{bmatrix}
\]

to

\[
\begin{bmatrix} \log c + k0 \\ \log c + k1 \\ \log c + k2 \\ \log c + k3 \end{bmatrix} = \log c \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + k \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} = \log cv_1 + kv_2.
\]

The system is written as usual:

\[
\begin{bmatrix} v_1 \cdot v_1 & v_1 \cdot v_2 \\ v_2 \cdot v_1 & v_2 \cdot v_2 \end{bmatrix} \begin{bmatrix} \log c \\ k \end{bmatrix} = \begin{bmatrix} b \cdot v_1 \\ b \cdot v_2 \end{bmatrix}
\]

Plug in numbers:

\[
\begin{bmatrix} 4 & 6 \\ 6 & 14 \end{bmatrix} \begin{bmatrix} \log c \\ k \end{bmatrix} = \begin{bmatrix} \log 1.3 + \log 1.5 + \log 1.7 \\ \log 1.3 + 2\log 1.5 + 3\log 1.7 \end{bmatrix}
\]

It remains to solve this system.