
20.1. Starting Example

Find eigenvalues and eigenvectors for

\[ A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \]

The characteristic polynomial is

\[ \det(A - \lambda I_3) = \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & 1 - \lambda \end{vmatrix} = (-\lambda)^2(1 - \lambda). \]

It has roots 0 and 1. Find eigenvectors corresponding to 0:

\[ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \]

The solution is \( x_2 = x_3 = 0 \), so

\[ x = \begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \]

This vector \( v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \) is the only (linearly independent) eigenvector corresponding to \( \lambda = 0 \). So algebraic multiplicity of this eigenvalue is two, and geometric multiplicity is one.

Find eigenvectors corresponding to \( \lambda = 1 \):

\[ \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \]

Write this as \(-x_1 + x_2 = 0, \ -x_2 = 0\), so \( x_1 = x_2 = 0 \), and

\[ x = \begin{bmatrix} 0 \\ 0 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \]

This vector \( v_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \) is the only (linearly independent) eigenvector corresponding to \( \lambda = 1 \). So both algebraic and geometric multiplicities of this eigenvalue is one.

20.2. General Theory

For each eigenvalue, its geometric multiplicity is always less than or equal to its algebraic multiplicity. The quantity of eigenvectors cannot exceed its multiplicity as a root. If for at least one eigenvalue these multiplicities are not equal, then there are not enough eigenvectors to form a basis (eigenbasis). In this case, the matrix is called defective. For example, the matrix above is defective. It has only two (linearly independent) eigenvectors: \( v_1 \) and \( v_2 \). They do not form a basis, because some vectors, for example \([0, 1, 0]^T\), are not expressible as their linear combinations.
20.3. Example of a Non-Defective Matrix

Let 

\[ A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \]

Then 

\[
\det(A - \lambda I_3) = \begin{vmatrix} 3 - \lambda & 0 & 0 \\ 0 & 2 - \lambda & 1 \\ 0 & 1 & 2 - \lambda \end{vmatrix}
\]

\[ = (3 - \lambda)(2 - \lambda)^2 - 1 - \lambda(\lambda - 3)(\lambda - 1) = -\lambda(3 - \lambda) \]

There are two eigenvalues, \( \lambda = 3 \) and \( \lambda = 1 \). The algebraic multiplicity of \( \lambda = 3 \) is two, and that of \( \lambda = 1 \) is one. Eigenvectors for \( \lambda = 3 \):

\[
\begin{bmatrix} 3 - \lambda & 0 & 0 \\ 0 & 2 - \lambda & 1 \\ 0 & 1 & 2 - \lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\]

Write this as \( x_2 = x_3 \). Every vector which satisfies this is an eigenvector for \( \lambda = 3 \) (of course, if it is not the zero vector). So 

\[
x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}
\]

So there are two linearly independent eigenvectors:

\[ v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \]

Let us find eigenvectors for \( \lambda = 1 \):

\[
\begin{bmatrix} 3 - \lambda & 0 & 0 \\ 0 & 2 - \lambda & 1 \\ 0 & 1 & 2 - \lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\]

So we have: \( 2x_1 = 0, \quad x_2 + x_3 = 0 \). Then 

\[
x = \begin{bmatrix} 0 \\ x_2 \\ -x_2 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}
\]

This is the only eigenvector corresponding to \( \lambda = 1 \):

\[ v_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \]

Therefore, geometric multiplicity of \( \lambda = 3 \) is two, and that of \( \lambda = 1 \) is one, same as their algebraic multiplicities. Therefore, this matrix is NOT defective.