
21.1. Example

Let

\[ A = \begin{bmatrix} 0 & -1 \\ 2 & 3 \end{bmatrix} \]

There are two eigenvalues, \( \lambda_1 = 1 \) and \( \lambda_2 = 2 \). Eigenvector corresponding to 1: \( v_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \)

Eigenvector corresponding to 2: \( v_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \).

Let \( x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = y_1 v_1 + y_2 v_2 \). Since \( v_1 \) and \( v_2 \) are linearly independent, they form a basis, so any vector \( x \) can be expressed as their linear combination. Then \( x = Sy \), where

\[ S = [v_1 | v_2] = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \]

is a matrix composed from column eigenvectors. We have:

\[ Ax = y_1 \lambda_1 v_1 + y_2 \lambda_2 v_2 = [v_1 | v_2] \begin{bmatrix} \lambda_1 y_1 \\ \lambda_2 y_2 \end{bmatrix} = S \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = SDy, \]

where

\[ D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \]

is the diagonal matrix composed from eigenvalues. Then we have:

\[ Ax = SDy, \text{ but also } Ax = ASy. \]

Then \( SDy = ASy \). Since these matrices act in the same way on any vector \( y \in \mathbb{R}^2 \), they are equal:

\[ SD = AS, \text{ so } A = SDS^{-1} \]

21.2. General Theory

Assume \( A \) has an eigenbasis, so it is not defective: for any eigenvalue, its algebraic and geometric multiplicities are the same. Then combine these eigenvectors \( v_1, \ldots, v_n \) into a matrix \( S = [v_1 | \ldots | v_n] \), and write eigenvalues in a diagonal matrix

\[ D = \begin{bmatrix} \lambda_1 & 0 & 0 & \ldots \\ 0 & \lambda_2 & 0 & \ldots \\ 0 & 0 & \lambda_3 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \]

Then

\[ A = SDS^{-1}. \]

Such (non-defective) matrix \( A \) is called diagonalizable. If it is defective, then all of the above does not work.
21.3. Raising Matrix to a Power

Calculate $A^{10}$ in the example above. Note that $A^2 = (SDS^{-1})(SDS^{-1}) = SD^2S^{-1}$, because we can cancel $S$ and $S^{-1}$ adjacent to each other. If they are not adjacent, then we cannot cancel them out, because the matrix product is not commutative! Similarly, $A^{10} = SD^{10}S^{-1}$. We have:

$$S^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}, \quad D^{10} = \begin{bmatrix} 1 & 0 \\ 0 & 2^{10} \end{bmatrix}$$

Therefore,

$$A^{10} = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2^{10} \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 2^{10} \\ -1 & -2 \cdot 2^{10} \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 2 - 2^{10} & 1 - 2^{10} \\ -2 + 2 \cdot 2^{10} & -1 + 2 \cdot 2^{10} \end{bmatrix}$$

21.4. Complex Eigenvalues and Eigenvectors

Let

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Its characteristic polynomial is

$$\begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1.$$ 

It does not have real roots, but it has complex roots $\pm i$. This matrix has complex eigenvalues, and also complex eigenvectors: for example,

$$\begin{bmatrix} i \\ 1 \end{bmatrix}$$

is an eigenvector for $\lambda = i$. 

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