
24.1. Formulation of the Problem

Assume there are only three airports in the US: Seattle-Tacoma (SEA), John F. Kennedy in New York (JFK), and Los Angeles International Airport (LAX). Each day, 30% of planes in SEA fly to LAX and 10% fly to JFK. The remaining 60% stay in SEA. Moreover, 10% of planes in JFK fly to LAX and 10% to SEA, the remaining 80% stay in New York. Finally, 10% of LAX planes fly to SEA and 20% to JFK, and 70% stay there. What is the limiting distribution, and how fast does it converge?

24.2. Mathematical Description

Suppose that today we had the fraction $x_1$ planes of the total quantity in SEA, $x_2$ in JFK, $x_3$ in LAX. Then tomorrow we will have 60% $x_1 + 10% x_2 + 10% x_3$ planes in SEA, 80% $x_2 + 10% x_1 + 20% x_3$ planes in JFK, 70% $x_3 + 30% x_1 + 10% x_2$ planes in LAX. So tomorrow’s distribution will be

$$A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \text{ where } A = \begin{bmatrix} 0.6 & 0.1 & 0.1 \\ 1 & 0.8 & 0.2 \\ 0.3 & 0.1 & 0.7 \end{bmatrix}$$

is the transition matrix. If $x(0)$ is the initial distribution, then $x(1) = Ax(0)$ is the distribution next day, $x(2) = Ax(1) = A^2 x(0)$ is the distribution the day after, etc. We would like to see how the vector $A^n x(0)$ behaves as $n \to \infty$.

24.3. The Eigenvalue Problem

This matrix has three eigenvalues:

$$\lambda_1 = 1, \  \lambda_2 = 0.6, \  \lambda_3 = 0.5.$$

Corresponding eigenvectors:

$$u_1 = \begin{bmatrix} 4 \\ 9 \\ 7 \end{bmatrix}, \  u_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \  u_3 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

We have:

$$Au_1 = u_1, \  Au_2 = 0.6 u_2, \  Au_3 = 0.5 u_3.$$ 

24.4. Finishing the Problem

These three vectors form an eigenbasis, and we can express the initial condition $x(0)$ as their linear combination:

$$x(0) = y_1 u_1 + y_2 u_2 + y_3 u_3.$$ 

To do this, we must solve the system

$$\begin{bmatrix} 4 & 0 & 1 \\ 9 & 1 & 1 \\ 7 & -1 & -2 \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$
So
\[ A^n u_1 = u_1, \quad A^n u_2 = 0.6^n u_2, \quad A^n u_3 = 0.5^n u_3. \]

Therefore,
\[ x(n) = A^n x(0) = y_1 u_1 + y_2 0.6^n u_2 + y_3 0.5^n u_3 \rightarrow y_1 u_1 \]
as \( n \to \infty \). But we do not need to find \( y_1 \) to calculate \( y_1 u_1 \), the limiting distribution. Indeed, this distribution must be proportional to \( u_1 \) and the sum of its elements (sum of shares of the quantity of planes) must be equal to one. Therefore, this is
\[
    w = \frac{1}{4 + 9 + 7} \begin{bmatrix} 4 \\ 9 \\ 7 \end{bmatrix} = \begin{bmatrix} 20\% \\ 45\% \\ 35\% \end{bmatrix}
\]

This is a stationary distribution, because \( Aw = w \): if today we have distribution \( w \), tomorrow it will be the same. And this is a limiting distribution, regardless of the initial distribution. The rate of convergence is \( 0.6^n \). We skip \( 0.5^n \) because it converges faster than \( 0.6^n \). We are interested in the rate (something to the power \( n \)), not the constant multiples, so we do not need to calculate \( y_2 \) and \( y_3 \).

### 24.5. Short Method

Find all eigenvalues. The unit eigenvalue will be among them. Find an eigenvector corresponding to this unit eigenvalue. Divide it by the sum of its entries, to make this sum equal to one. Then this will be the stationary (and limiting) distribution. From other eigenvalues, select the one \( \lambda \) such that \( |\lambda| \) is closest to one. Because if \( |\lambda_1| < |\lambda_2| \), then \( \lambda_1^n \) converges to zero faster than \( \lambda_2^n \). Then this \( \lambda \) will give the rate of convergence: \( \lambda^n \).

You do NOT need to find \( y_1, y_2, y_3 \) to do this. Because the stationary distribution does NOT depend on the initial distribution. The same is true for the rate of convergence.

### 24.6. Answer

The stationary distribution is
\[
    u = \begin{bmatrix} 20\% \\ 45\% \\ 35\% \end{bmatrix}
\]

The rate of convergence is \( 0.6^n \).