
4.1. Basic Concepts

An \( m \times n \)-table of real numbers

\[
A = \begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{bmatrix}
\]

is called an \( m \times n \)-matrix. It has \( m \) rows and \( n \) columns. The numbers \( a_{ij} \) are called elements or entries. There are \( mn \) of them. Each entry \( a_{ij} \) is at the intersection of the \( i \)th row and the \( j \)th column, \( 1 \leq i \leq m, \ 1 \leq j \leq n \).

We write: \( A = (a_{ij})_{1 \leq i \leq m, 1 \leq j \leq n} \). We also write: \([\text{number of rows}] \times [\text{number of columns}]\)-matrix.

For example, let

\[
A = \begin{bmatrix}
1 & 2 \\
3 & 4 \\
5 & 6
\end{bmatrix}
\]

Then \( n = 3, \ m = 2, \) and \( a_{11} = 1, \ a_{12} = 2, \ a_{32} = 6, \) and \( a_{33} \) is undefined.

We consider vectors \( x \in \mathbb{R}^n \) as \( n \times 1 \)-matrices and write them as columns:

\[
x = \begin{bmatrix}
x_1 \\
x_2 \\
\vdots \\
x_n
\end{bmatrix}
\]

4.2. Addition and Scalar Multiplication

We can add matrices of the same size by just adding their respective elements. For example:

\[
\begin{bmatrix}
1 & 2 \\
3 & 4 \\
5 & 6
\end{bmatrix} + \begin{bmatrix}
1 & 0 \\
1 & -1 \\
2 & 0
\end{bmatrix} = \begin{bmatrix}
2 & 2 \\
4 & 3 \\
7 & 6
\end{bmatrix}
\]

If these two matrices have different numbers of rows or different numbers of columns, then their sum does not make sense. Also, we can multiply a matrix by a scalar just by multiplying each entry by this scalar. Example:

\[
3 \begin{bmatrix}
1 & 2 \\
3 & 4 \\
0 & -1
\end{bmatrix} = \begin{bmatrix}
3 & 6 \\
9 & 12 \\
0 & -3
\end{bmatrix}
\]

4.3. Matrix Transposition

Let \( A \) be a matrix. If we interchange rows and columns, then we get the matrix \( B \), which is called the transpose of \( A \), and is denoted by \( B = A^T \). The rows of \( B \) are the columns of \( A \), and the columns of \( B \) are the rows of \( A \). If \( A = (a_{ij}) \) and \( B = (b_{ij}) \), then \( b_{ij} = a_{ji} \) for all \( i, j \).

For example,

\[
A = \begin{bmatrix}
1 & 2 \\
3 & 4 \\
5 & 6
\end{bmatrix} \Rightarrow B = A^T = \begin{bmatrix}
1 & 3 & 5 \\
2 & 4 & 6
\end{bmatrix}
\]
A matrix $A$ such that $A^T = A$ is called symmetric. Example: $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$

4.4. Identity Matrix

A matrix with $m = n$ (the same number of rows and columns) is called a square matrix. Its main diagonal is formed by elements $a_{11}, a_{22}, \ldots, a_{nn}$. For example, the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

is $3 \times 3$ diagonal with 1, 5, 9 on the main diagonal.

One example of a square matrix is the identity matrix $I_n$, which is an $n \times n$-matrix with units on the main diagonal and zeros elsewhere. It acts as a unity for matrix multiplication. Indeed, $I_n x = x$ for $x \in \mathbb{R}^n$. Let us check this for $n = 3$:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

In other words, when you apply this matrix to vectors of appropriate dimension, it does not change them. This is why it is called the identity matrix.

The same applies to matrices: if $A$ is $m \times n$-matrix, then

$$I_m A = AI_n = A.$$

4.5. Dot Product

For vectors $x, y \in \mathbb{R}^n$,

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

we have: $x^T = [x_1 \ x_2 \ \ldots \ \ x_n]$ is an $1 \times n$-matrix, and $y$ is an $n \times 1$ matrix. So we can multiply them as matrices and get a $1 \times 1$-matrix, i.e. a number:

$$x^T y = x_1 y_1 + \ldots + x_n y_n.$$

This is called the dot product of $x$ and $y$ and is denoted by $x \cdot y$. Example:

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 1 \cdot 1 + 2 \cdot 1 + 3 \cdot 1 = 6.$$

The quantity

$$|x| = ||x|| = \sqrt{x_1^2 + \ldots + x_n^2} = \sqrt{x^T x} = \sqrt{x \cdot x}$$

is called the norm, length or magnitude of $x$. Example:

$$x = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \implies ||x|| = 2.$$
4.6. Property of matrix transposition

For any two matrices $A, B$ with appropriate dimensions, 

$$(AB)^T = B^T A^T.$$ 

Proof. We shall take $ij$th element of the left-hand side and show that it is equal to the $ij$-th element of the right-hand side. The $ij$-th element of $(AB)^T$ = the $ji$-th element of $AB$ = the dot product of $j$th row of $A$ and $i$th column of $B$ = the dot product of $j$th column of $A^T$ and $i$th row of $B^T$ = the $ij$th element of $B^T A^T$. 
