7. Linear Independence. April 15, 2013

7.1. Definition

If we have vectors \( v_1, \ldots, v_m \in \mathbb{R}^n \), then they are linearly independent if the system of equations

\[
x_1 v_1 + \ldots + x_m v_m = 0
\]

has only the trivial solution: \( x_1 = \ldots = x_m = 0 \). We can write it in the form \( Ax = 0 \), where \( A = [v_1 v_2 \ldots v_m] \) is an \( n \times m \)-matrix.

7.2. Example

The vectors \( v_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \) and \( v_2 = \begin{bmatrix} 2 \\ 4 \end{bmatrix} \) are linearly independent. Indeed, solve the following system:

\[
x_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.
\]

We have:

\[
\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},
\]

so

\[
A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.
\]

Solve this system:

\[
[A|0] \Rightarrow \begin{bmatrix} 1 & 2 & 0 \\ 0 & -2 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.
\]

So it has only trivial solution \( x_1 = x_2 = 0 \).

7.3. Geometric Meaning

If two vectors \( v_1, v_2 \) are linearly dependent, then we have:

\[
x_1 v_1 + x_2 v_2 = 0,
\]

and at least one of \( x_1, x_2 \) is nonzero. Assume \( x_1 \neq 0 \), then

\[
v_1 = -\frac{x_2}{x_1} v_2.
\]

So these vectors are parallel, and one of them is a scalar multiple of the other: for example, \( v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \) and \( v_2 = \begin{bmatrix} 2 \\ 4 \end{bmatrix} \).

If three vectors are linearly dependent:

\[
x_1 v_1 + x_2 v_2 + x_3 v_3 = 0,
\]

where at least one of \( x_1, x_2, x_3 \) is not zero, say, \( x_1 \neq 0 \), then

\[
v_1 = -\frac{x_2}{x_1} v_2 - \frac{x_3}{x_1} v_3,
\]

so \( v_1 \) lies on the plane generated by \( v_2 \) and \( v_3 \).
7.4. Linear Independence and Nonsingular Matrices

Assume an \( n \times n \)-matrix \( A \) is nonsingular. Let \( v_1, \ldots, v_n \) be its column vectors. Then the system \( Ax = 0 \) can be reduced to \( [I_n|0] \), so its only solution is \( x_1 = \ldots = x_n = 0 \). And these vectors are linearly independent. So \( A \) is invertible (=nonsingular) if and only if its columns are linearly independent.

7.5. Solving Systems by Matrix Inverses

Assume \( Ax = b \) is the system of equations, where \( A \) is an \( n \times n \)-matrix, \( x, b \in \mathbb{R}^n \). Assume \( A \) is invertible. We proved last time that \( AA^{-1} = I_n \). But in this special case (multiplying a matrix by its own inverse), the matrix multiplication is commutative, and so \( A^{-1}A = I_n \). Then we have:

\[
    Ax = b \Rightarrow A^{-1}Ax = A^{-1}b \Rightarrow I_nx = A^{-1}b \Rightarrow x = A^{-1}b.
\]

This is not a very efficient way to solve linear systems (compared to row reduction), but theoretically it exists.