Name ___________________________________  Student ID #__________________

- Your exam should consist of this cover sheet, followed by 7 problems. Check that you have a complete exam.

- Unless otherwise indicated, show all your work and justify your answers.

- Unless otherwise indicated, your answers should be exact values rather than decimal approximations. For example, $\frac{\pi}{4}$ is an exact answer and is preferable to 0.7854.

- You may use a scientific calculator and one double-sided 8.5×11-inch sheet of handwritten notes. All other electronic devices, including graphing or programmable calculators, and calculators which can do calculus, are forbidden.

- The use of headphones or earbuds during the exam is not permitted.

- **Show your work**, unless instructed otherwise.

- If you need more space, write on the back and indicate this. If you still need more space, raise your hand and I’ll give you some extra paper to staple onto the back of your test.

- Academic misconduct will guarantee a score of zero on this exam. **DO NOT CHEAT.**

- Turn your cell phone OFF and put it AWAY for the duration of the exam.

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<th>Problem</th>
<th>Points</th>
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<td>Total:</td>
<td>70</td>
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1. Give an example of:
   
   (a) (4 points) a $2 \times 2$ defective matrix
   
   **Solution:**
   
   $\begin{bmatrix}
   1 & 1 \\
   0 & 1 
   \end{bmatrix}$
   
   (b) (3 points) $3 \times 3$-matrices $A$ and $B$ such that $\det(A + B) \neq \det(A) + \det(B)$
   
   **Solution:**
   
   $A = B = I_3 = \begin{bmatrix}
   1 & 0 & 0 \\
   0 & 1 & 0 \\
   0 & 0 & 1
   \end{bmatrix}$ then $A + B = \begin{bmatrix}
   2 & 0 & 0 \\
   0 & 2 & 0 \\
   0 & 0 & 2
   \end{bmatrix}$ so $\det(A + B) = 8$, $\det(A) = \det(B) = 2$.
   
   (c) (3 points) a matrix with rank 2 and nullity 3
   
   **Solution:**
   
   $A = \begin{bmatrix}
   1 & 0 & 0 & 0 & 0 \\
   0 & 1 & 0 & 0 & 0 \\
   0 & 0 & 0 & 0 & 0
   \end{bmatrix}$
2. (10 points) Find the eigenvalues and eigenvectors of the matrix

\[
A = \begin{bmatrix}
2 & 4 & 4 \\
0 & 1 & -1 \\
0 & 1 & 3
\end{bmatrix}
\]

Indicate the algebraic and geometric multiplicity of each eigenvalue.

**Solution:** The characteristic polynomial:

\[
\det(A - \lambda I_3) = \begin{vmatrix}
2 - \lambda & 4 & 4 \\
0 & 1 - \lambda & -1 \\
0 & 1 & 3 - \lambda
\end{vmatrix} = (2 - \lambda) \begin{vmatrix}
1 - \lambda & -1 \\
1 & 3 - \lambda
\end{vmatrix} =
\]

\[
(2 - \lambda)[(1 - \lambda)(3 - \lambda) + 1] = (2 - \lambda)(\lambda^2 - 4\lambda + 4) = -(\lambda - 2)^3.
\]

So the only eigenvalue is \(\lambda = 2\), and its algebraic multiplicity is three. Corresponding eigenvectors:

\[
(A - 2I_3)x = 0 \Rightarrow \begin{bmatrix}
0 & 4 & 4 \\
0 & -1 & -1 \\
0 & 1 & 1
\end{bmatrix} \Rightarrow x_2 + x_3 = 0.
\]

So there are two linearly independent eigenvectors:

\[
v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.
\]

The geometric multiplicity is two.
3. (10 points) Suppose you input $x, y$ and measure the output $z(x, y)$. Approximate $z = \alpha x + \beta y + \gamma$ by least squares method. Set the $3 \times 3$ system, but DO NOT SOLVE it. You have the following table of measurements:

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
<th>$z$</th>
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</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>-1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>4</td>
</tr>
</tbody>
</table>

**Solution:** We have:

$$
\begin{align*}
\alpha 0 + \beta 0 + \gamma &= 0 \\
\alpha 1 + \beta 0 + \gamma &= 3 \\
\alpha 0 + \beta 1 + \gamma &= 2 \\
\alpha (-1) + \beta 1 + \gamma &= 1 \\
\alpha 1 + \beta 2 + \gamma &= 4
\end{align*}
$$

Let us find $\alpha, \beta, \gamma$ such that the distance between $b = \begin{bmatrix} 0 \\ 3 \\ 2 \\ 1 \\ 4 \end{bmatrix}$ and $\alpha v_1 + \beta v_2 + \gamma v_3$, where

$$
\begin{align*}
v_1 &= \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \\
v_2 &= \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 2 \end{bmatrix} \\
v_3 &= \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}
\end{align*}
$$

is minimal. To this end, solve the system:

$$
\begin{bmatrix} 
v_1 \cdot v_1 & v_1 \cdot v_2 & v_1 \cdot v_3 \\
v_2 \cdot v_1 & v_2 \cdot v_2 & v_2 \cdot v_3 \\
v_3 \cdot v_1 & v_3 \cdot v_2 & v_3 \cdot v_3
\end{bmatrix}
\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} =
\begin{bmatrix} 
b \cdot v_1 \\
b \cdot v_2 \\
b \cdot v_3
\end{bmatrix}
$$

Write it with numbers:

$$
\begin{bmatrix} 
3 & 1 & 1 \\
1 & 6 & 4 \\
1 & 4 & 5
\end{bmatrix}
\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} =
\begin{bmatrix} 6 \\ 11 \\ 10 \end{bmatrix}
$$
4. Consider the matrix

\[
A = \begin{bmatrix}
1 & 3 \\
2 & 7 \\
1 & 5 \\
\end{bmatrix}
\]

(a) (5 points) Find its rank and a basis for its range.

**Solution:** Since the two columns are linearly independent (they are not proportional), they serve as a basis for the range of \(A\). Its dimension (=the rank of \(A\)) is two.

(b) (5 points) Find its nullity and a basis for its nullspace.

**Solution:** The two columns \(v_1, v_2\) are linearly independent, so \(x_1 v_1 + x_2 v_2 = 0 \iff Ax = 0\) has only the trivial solution (zero vector). So the nullspace consists of the single zero vector, its dimension (=the nullity of \(A\)) is zero, and it does not have a basis.
5. (10 points) Consider the matrix
\[ A = \begin{bmatrix} 5 & 2 \\ -2 & 0 \end{bmatrix} \]

Find a diagonal matrix \( D \) and a nonsingular matrix \( S \) such that \( A = SDS^{-1} \).

**Solution:** Characteristic polynomial: \( \lambda^2 - 5\lambda + 4 \). Eigenvalues: \( \lambda_{1,2} = 1, 4 \). Corresponding eigenvectors:

\[ v_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \]

So

\[ D = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}, \quad S = \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix} \]
6. (10 points) Consider a Markov chain with two states. The transition probability from the first to the second state is 1/3, and the transition probability from the second to the first state is 1/2. What is the limiting distribution? What is the speed of convergence?

**Solution:** Transition matrix:

\[
A = \begin{bmatrix} \frac{2}{3} & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} \end{bmatrix}
\]

Characteristic polynomial:

\[
\lambda^2 - \frac{7}{6} \lambda + \frac{1}{6}.
\]

Eigenvalues: \( \lambda_1 = 1, \lambda_2 = 1/6 \). The eigenvector corresponding to \( \lambda_1 = 1 \) is \([1/2, 1/3]^T\). Normalize it: \([3/5, 2/5]\). The speed of convergence: \((1/6)^n\).
7. (10 points) Find a substitution $x = Qy$ that diagonalizes the given quadratic form, where $Q$ is orthogonal. And write the form in new coordinates.

$$q(x) = x_1^2 + x_2^2 - x_1 x_2$$

**Solution:** The matrix:

$$A = \begin{bmatrix} 1 & -1/2 \\ -1/2 & 1 \end{bmatrix}$$

Eigenvalues: $\lambda_1 = 3/2$, $\lambda_2 = 1/2$. Eigenvectors: $v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. This is an orthogonal eigenbasis. Normalize it:

$$w_1 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}, \quad w_2 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

These columns form $Q$. So we have:

$$x_1 = \frac{1}{\sqrt{2}} y_1 + \frac{1}{\sqrt{2}} y_2, \quad x_2 = -\frac{1}{\sqrt{2}} y_1 + \frac{1}{\sqrt{2}} y_2.$$ 

And

$$q(y) = \lambda_1 y_1^2 + \lambda_2 y_2^2 = \frac{3}{2} y_1^2 + \frac{1}{2} y_2^2.$$