Consider the vectors
\[ v_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \]

(a) Are these vectors linearly dependent?

(b) Describe the set of vectors representable as linear combinations of \( v_1, v_2, v_3 \). This is called the *span* of \( v_1, v_2, v_3 \). You do NOT have to provide concrete coefficients of linear combinations.

**Solution.**

(a) Let us form a matrix
\[
A = \begin{bmatrix}
0 & 1 & 1 \\
1 & 0 & 1 \\
1 & 1 & 0
\end{bmatrix}
\]
and find whether it is nonsingular, i.e. whether it reduces to the identity matrix. We get:

\[
A \Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]

Thus, the vectors are **linearly independent**. The system \( Ax = 0 \), which is \( x_1 v_1 + x_2 v_2 + x_3 v_3 = 0 \), has only the zero solution.

(b) A vector \( b \in \mathbb{R}^3 \) belongs to the span if and only if the system \( Ax = b \) has a solution. Let us reduce the matrix \([A|b]\) to reduced echelon form: it will be \([I_3|c] \), where \( c \) is some other vector. So this system always has a solution. Any vector \( b \in \mathbb{R}^3 \) lies in the span of \( v_1, v_2, v_3 \). Answer: \( \mathbb{R}^3 \)