A Fourier series for the function $f : [0, 2\pi] \to \mathbb{R}$ is an expression of the form

$$f(x) = \frac{a_0}{2} + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \ldots$$

Here, we have:

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos(nx) dx, \quad n \geq 0, \quad b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin(nx) dx, \quad n \geq 1.$$

**Parseval’s Identity.** The functions $1/2, \cos x, \sin x, \cos 2x, \sin 2x, \ldots$ are orthogonal to each other, and so

$$\int_0^{2\pi} f^2(x) dx = \int_0^{2\pi} \left[ \frac{a_0}{2} + a_1 \cos x + b_1 \sin x + \ldots \right] \left[ \frac{a_0}{2} + a_1 \cos x + b_1 \sin x + \ldots \right] dx = \int_0^{2\pi} \left( \frac{a_0}{2} \right)^2 dx + a_1^2 \int_0^{2\pi} \cos^2 x dx + b_1^2 \int_0^{2\pi} \sin^2 x dx + \ldots$$

All cross-terms vanish, because the functions are orthogonal. Now, we have:

$$\int_0^{2\pi} \cos^2 nx dx = \int_0^{2\pi} \sin^2 nx dx = \pi.$$

Therefore, the last infinite sum is equal to

$$a_0^2 \pi^2 + a_1^2 \pi^2 + b_1^2 \pi^2 + \ldots = \pi \left[ \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \right].$$

Thus, we have:

$$\frac{1}{\pi} \int_0^{2\pi} f^2(x) dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2).$$

This is called *Parseval’s identity*. It is similar to Pythagoras’ theorem, but in infinite dimension.

**Odd and even functions.** A function $f$ is called *odd* if $f(-x) = -f(x)$. A function $f$ is called *even* if $f(-x) = f(x)$. For an even function, $\int_{-\pi}^{\pi} f(x) dx = 2 \int_0^{2\pi} f(x) dx$. For an odd function, $\int_{-\pi}^{\pi} f(x) dx = 0$.

**Example.** We can consider Fourier series on $[-\pi, \pi]$ rather than $[0, 2\pi]$. Let

$$f(x) = \begin{cases} x, & -\pi < x < \pi; \\ 0, & x = \pm \pi \end{cases}$$

Then

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx = 0, \quad n \geq 0,$$

because $f$ is an odd function, $\cos nx$ is even, so their product is odd. Also,

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx = \frac{2(-1)^{n+1}}{n}, \quad n \geq 1.$$

This is done using integration by parts. Finally,

$$\int_{-\pi}^{\pi} f^2(x) dx = \int_{-\pi}^{\pi} x^2 dx = \frac{2x^3}{3} \bigg|_{x=-\pi}^{x=\pi} = \frac{2\pi^3}{3}.$$
So we have, plugging into Parseval’s identity:

\[ \sum_{n=1}^{\infty} \left( \frac{2(-1)^{n+1}}{n} \right)^2 = \frac{1}{\pi^3} \frac{2\pi^3}{3}, \]

which can be written as

\[ \sum_{n=1}^{\infty} \frac{4}{n^2} = \frac{2\pi^2}{3} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \]

This is the famous *Basel Problem*, resolved by the young Leonard Euler.

**Fourier Series on different intervals** For \([0, 2L]\) or \([-L, L]\) instead of \([-\pi, \pi]\), we can also consider Fourier series:

\[ f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos \frac{\pi nx}{L} + b_n \sin \frac{\pi nx}{L} \right). \]

Here,

\[ a_n := \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{\pi nx}{L} \, dx, \quad n \geq 0; \quad b_n := \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{\pi nx}{L} \, dx, \quad n \geq 1. \]

By the way, the same function may have different Fourier series decomposition on different intervals. Let

\[ f(x) = \begin{cases} 1, & 0 \leq x \leq \pi; \\ 0, & \text{else} \end{cases} \]

Then on \([0, 2\pi]\) the Fourier series decomposition was found in the last lecture:

\[ f(x) = \frac{1}{2} + \frac{2}{\pi} \sin x + \frac{2}{3\pi} \sin(3x) + \ldots \]

And on \([0, \pi]\) this function is identically one, so the decomposition is simple: \( f = 1 \).