
Now, consider the wave equation

$$u_{tt} = a^2 u_{xx}$$

with initial conditions

$$u|_{t=0} = f(x), \quad u_t|_{t=0} = g(x),$$

and boundary conditions

$$u_x|_{x=0} = 0, \quad u_x|_{x=L} = 0.$$  

Separation of variables: let $$u(t, x) = X(t)T(t),$$ we have:

$$T''(t)X(x) = a^2T(t)X''(x) \Rightarrow \frac{T''(t)}{a^2T(t)} = \frac{X''(x)}{X(x)}.$$  

The left-hand side depends only on $$t$$, and the right-hand side depends only on $$x$$. So they are both constant, equal to $$-\lambda$$. We have:

$$X''(x) + \lambda X(x) = 0.$$  

Since $$u_x|_{x=0} = 0$$, this means $$X'(0) = 0$$. Since $$u_x|_{x=L} = 0$$, this means $$X'(L) = 0$$. Therefore, we have the boundary value problem:

$$X''(x) + \lambda X(x) = 0, \quad X'(0) = X'(L) = 0.$$  

This BVP has nonzero solutions if $$\lambda = (\pi n/L)^2, \quad n = 0, 1, 2, \ldots,$$ and $$X(x) = \cos(\pi n x/L).$$ This includes the case $$n = 0$$, when $$\lambda = 0$$ and $$X(x) = 1$$. Now, solve for $$T(t)$$.

**Case 1.** $$n = 0$$. Then $$\lambda = 0$$, and we have: $$T''(t) = 0, \quad T'(t) = C_1, \quad T(t) = C_1 t + C_2$$. Building block:

$$u(t, x) = C_1 t + C_2, \quad n = 0.$$  

**Case 2.** $$n = 1, 2, 3, \ldots$$. Then

$$T''(t) + a^2 \lambda T(t) = 0, \quad T''(t) + \left(\frac{\pi an}{L}\right)^2 T(t) = 0,$$

and the solution is

$$T(t) = C_1 \cos\left(\frac{\pi an}{L}t\right) + C_2 \sin\left(\frac{\pi an}{L}t\right).$$

Building block:

$$u(t, x) = \left[ C_1 \cos\left(\frac{\pi an}{L}t\right) + C_2 \sin\left(\frac{\pi an}{L}t\right) \right] \cos\left(\frac{\pi nx}{L}\right), \quad n \geq 1.$$  

Each of the blocks satisfies the wave equation and the boundary conditions, but not the initial conditions. Sum those building blocks to satisfy the initial conditions:

$$u(t, x) = \sum_{n=1}^{\infty} \left[ a_n \cos\left(\frac{\pi an}{L}t\right) + b_n \sin\left(\frac{\pi an}{L}t\right) \right] \cos\left(\frac{\pi nx}{L}\right) + \frac{b_0}{2} t + \frac{a_0}{2}.$$  

We have:

$$f(x) = u(0, x) = \sum_{n=1}^{\infty} a_n \cos\left(\frac{\pi nx}{L}\right) + \frac{a_0}{2}, \quad g(x) = u_t(0, x) = \sum_{n=1}^{\infty} b_n \frac{\pi an}{L} \cos\left(\frac{\pi nx}{L}\right) + \frac{b_0}{2}.$$  

We can find the coefficients:

$$a_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{\pi nx}{L}\right) dx, \quad n \geq 0,$$

$$b_n = \frac{2}{L} \int_0^L g(x) \cos\left(\frac{\pi nx}{L}\right) dx, \quad n \geq 1,$$

and from here we can find $$b_n.$$