Lecture 22. Wave Equation III. March 5, 2014

Consider an example of a boundary value problem for the wave equation:

$$u_{tt} = 9u_{xx},$$

with boundary conditions

$$u|_{x=0} = u|_{x=2\pi} = 0,$$

and initial conditions

$$u|_{t=0} = \sin(6x), \quad u_t|_{t=0} = \sin(2x) - \sin(4x).$$

Separation of variables: let $u(t, x) = X(x)T(t)$, we have:

$$T''(t)X(x) = 9T(t)X''(x) \Rightarrow \frac{T''(t)}{9T(t)} = \frac{X''(x)}{X(x)}.$$

The left-hand side depends only on $t$, and the right-hand side depends only on $x$. So they are both constant, equal to $-\lambda$. We have:

$$X''(x) + \lambda X(x) = 0.$$

Since $u(t, 0) = 0$, this means in terms of $X(x)$ that $X(0) = 0$. Similarly, $X(2\pi) = 0$. We have the boundary value problem

$$X''(x) + \lambda X(x) = 0, \quad X(0) = 0, \quad X(2\pi) = 0.$$

Solution:

$$\lambda = \left(\frac{\pi n}{2}\right)^2, \quad n = 1, 2, \ldots, X(x) = \sin\left(\frac{nx}{2}\right).$$

Solve for $T(t)$:

$$T''(t) + \left(\frac{3n}{2}\right)^2 T(t) = 0, \quad T(t) = C_1 \cos\left(\frac{3n}{2}t\right) + C_2 \sin\left(\frac{3n}{2}t\right).$$

Building block:

$$u(t, x) = \left[a_n \cos\left(\frac{3n}{2}t\right) + b_n \sin\left(\frac{3n}{2}t\right)\right] \sin\left(\frac{nx}{2}\right).$$

Each of the blocks satisfies the wave equation and the boundary conditions, but not the initial conditions. Sum those building blocks to satisfy also the initial conditions:

$$u(t, x) = \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{3n}{2}t\right) + b_n \sin\left(\frac{3n}{2}t\right)\right] \sin\left(\frac{nx}{2}\right).$$

We have:

$$u(0, x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{nx}{2}\right) = \sin(6x),$$

$$u_t(0, x) = \sum_{n=1}^{\infty} b_n \frac{3n}{2} \sin\left(\frac{nx}{2}\right) = \sin(2x) - \sin(4x).$$

We can find $a_n, b_n$ by comparing coefficients: the only nonzero coefficient in the first series is when $n = 12$, and $a_{12} = 1$. The only two nonzero coefficients in the second series are for $n = 4$ and $n = 8$. For $n = 4$, we have:

$$b_4 \frac{3 \cdot 4}{2} = 1 \Rightarrow b_4 = \frac{1}{6}.$$
And for \( n = 8 \), we have:
\[
b_8 \frac{3 \cdot 8}{2} = -1 \Rightarrow b_8 = -\frac{1}{12}.
\]
The final solution is
\[
u(t, x) = \cos(18t) \sin(6x) + \frac{1}{6} \sin(6t) \sin(2x) - \frac{1}{12} \sin(12t) \sin(4x)
\]