Lecture 25. Laplace Equation II. March 12, 2014

Consider the BVP for the Laplace equation \( \Delta u = u_{xx} + u_{yy} = 0 \) in the disc: \( x^2 + y^2 \leq a^2 \), where \( a \) is the radius of this disc. The boundary condition is \( u = f \) on the circle which is the boundary of the disc. We can rewrite the Laplace operator in polar coordinates as

\[
\Delta u(r, \theta) = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta \theta}.
\]

So we are given the following problem: find a function \( u(r, \theta) \), \( r \geq 0 \), which is \( 2\pi \)-periodic in \( \theta \), and satisfies the Laplace equation

\[
u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta \theta} = 0
\]

and the boundary condition \( u|_{r=a} = f(\theta) \).

Try separation of variables: satisfy the Laplace equation and the \( 2\pi \)-periodicity property (but not yet the boundary condition) using the function \( u(r, \theta) = R(r)\Theta(\theta) \). Then

\[
\frac{R''(r) + \frac{1}{r} R'(r)}{R(r)} = -\frac{\Theta''(\theta)}{\Theta(\theta)} = \lambda.
\]

So we have the following boundary value problem.

\[\Theta''(\theta) + \lambda \Theta(\theta) = 0, \Theta \text{ is } 2\pi\text{-periodic.}\]

**Solution.** \( \lambda = 0; \lambda = n^2, n = 1, 2, \ldots \); for \( \lambda = 0 \) we have: \( \Theta(\theta) = C \), for \( \lambda = n^2 \), \( n = 1, 2, \ldots \) we have: \( \Theta(\theta) = C_1 \cos nt + C_2 \sin nt \).

Now, the equation for \( R(r) \):

\[
R''(r) + \frac{1}{r} R'(r) = \frac{n^2}{r^2} R(r), \quad n = 0, 1, 2, \ldots
\]

**Case 1.** \( n = 1, 2, 3, \ldots \). Let us find its solutions in the form: \( R(r) = r^c \), where \( c \) is a real number. We have:

\[
c(c - 1)r^{c-2} + cr^{c-1} \frac{1}{r} = \frac{n^2}{r^2} r^c \quad \Rightarrow \quad c(c - 1) + c = n^2,
\]

so \( c^2 = n^2 \), and \( c = \pm n \), \( n \geq 1 \). So \( R(r) = C_1 r^n + C_2 r^{-n} \). But the function \( u(r, \theta) = R(r)\Theta(\theta) \) should be defined at \( r = 0 \). So the term \( r^{-n} \) cannot be here. We have: \( C_2 = 0 \), and \( R(r) = C_1 r^n \).

**Case 2.** \( n = 0 \). Then

\[
R'' + \frac{R'}{r} = 0 \quad \Rightarrow \quad r R'' + R' = 0 \quad \Rightarrow \quad (R' r)' = 0
\]

\[
\Rightarrow \quad R' r = C_1 \quad \Rightarrow \quad R = \frac{C_1}{r} \quad \Rightarrow \quad R = C_1 \log r + C_2.
\]

But \( R(r) \) should be defined at \( r = 0 \), so there is no term \( R(r) = C_1 \log r \) respectively. Write it as follows: For \( n = 0 \), we have: \( R(r) = C \), and for \( n \geq 1 \), we have: \( R(r) = C r^n \). These cases can be both incorporated into a single formula: \( R(r) = C r^n \), \( n \geq 0 \).

The building blocks:

\[u_n(r, \theta) = r^n(C_1 \cos(n\theta) + C_2 \sin(n\theta)), \quad n = 1, 2, \ldots ; u_0(r, \theta) = 1.\]

Therefore, to satisfy the boundary condition, we should sum them into a series:

\[
u(r, \theta) = c_0 + \sum_{n=1}^{\infty} [c_n \cos(n\theta) + d_n \sin(n\theta)] r^n.
\]
Find coefficients \( c_n, \ n = 0, 1, 2, \ldots; d_n, \ n = 1, 2, \ldots \) so that \( u(a, \theta) = f(\theta) \). Plug in \( r = a \):

\[
c_0 + \sum_{n=1}^{\infty} [c_n \cos(n\theta) + d_n \sin(n\theta)] a^n = f(\theta).
\]

Decompose \( f \) into Fourier series on \([0, 2\pi] \):

\[
f(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(n\theta) + b_n \sin(n\theta)).
\]

The coefficients are calculated using the usual rules:

\[
a_n := \frac{1}{\pi} \int_{0}^{2\pi} f(\theta) \cos(n\theta) d\theta, \quad b_n := \frac{1}{\pi} \int_{0}^{2\pi} f(\theta) \sin(n\theta) d\theta.
\]

Then

\[
c_0 = 2a_0, \quad c_n = a^{-n}a_n, \quad d_n = a^{-n}b_n, \quad n = 1, 2, \ldots
\]

**Example.** \( a = 1, \ f = y \). Then \( f = r \cos \theta = \cos \theta \) at the boundary of the disc, and \( u = r \cos \theta = y \). We could have written it immediately, since the function \( f(x, y) = y \) satisfies the Laplace equation.

**Exercise.** \( a = 2, \ f = 2 \cos^2 \theta \). Then \( f = \cos(2\theta) + 1 \), and

\[
u = 1 + \left(\frac{r}{2}\right)^2 \cos(2\theta).
\]