Lecture 4. The Eigenvalue Problem II. January 13, 2014

If $Ax = \lambda x$, where $\lambda$ is a number, and $x \neq 0$ is a vector, the number $\lambda$ is called an eigenvalue, and the vector $x$ - an eigenvector.

For an eigenvalue $\lambda$, its algebraic multiplicity is the multiplicity of $\lambda$ as a root of the characteristic polynomial. Its geometric multiplicity is the maximal number of linearly independent eigenvectors corresponding to it.

When at least one geometric multiplicity is smaller than the corresponding algebraic multiplicity, the matrix is called defective. For each eigenvalue, its geometric multiplicity is always less than or equal to its algebraic multiplicity. The quantity of eigenvectors cannot exceed its multiplicity as a root. If for at least one eigenvalue these multiplicities are not equal, then there are not enough eigenvectors to form a basis (eigenbasis). In this case, the matrix is called defective. Such matrices are not diagonalizable. So a matrix is either defective or diagonalizable.

Example 1. Find eigenvalues and eigenvectors for

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

The characteristic polynomial is

$$\det(A - \lambda I_3) = \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & 2 - \lambda \end{vmatrix} = (-\lambda)^2(2 - \lambda).$$

It has roots 0 and 2. Find eigenvectors corresponding to 0:

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The solution is $x_2 = x_3 = 0$, so

$$x = \begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

This vector $v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is the only (linearly independent) eigenvector corresponding to $\lambda = 0$. So algebraic multiplicity of this eigenvalue is two, and geometric multiplicity is one.

Find eigenvectors corresponding to $\lambda = 2$:

$$\begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Write this as $-2x_1 + x_2 = 0$, $-2x_2 = 0$, so $x_1 = x_2 = 0$, and

$$x = \begin{bmatrix} 0 \\ 0 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

This vector $v_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is the only (linearly independent) eigenvector corresponding to $\lambda = 1$. So both algebraic and geometric multiplicities of this eigenvalue is one.
This matrix is defective. It has only two (linearly independent) eigenvectors: \(v_1\) and \(v_2\). They do not form a basis, because some vectors, for example \([0, 1, 0]^T\), are not expressible as their linear combinations.

**Example 2.** Find eigenvalues and eigenvectors for

\[
A = \begin{bmatrix}
0 & 0 & 0 \\
0 & 1 & 1 \\
0 & 1 & 1
\end{bmatrix}
\]

The characteristic polynomial is

\[
\det(A - \lambda I_3) = \begin{vmatrix}
-\lambda & 0 & 0 \\
0 & 1 - \lambda & 1 \\
0 & 1 & 1 - \lambda
\end{vmatrix} = (-\lambda)(1 - \lambda)^2 - 1 = -\lambda(\lambda^2 - 2\lambda).
\]

It has roots 0 and 2. Find eigenvectors corresponding to 0:

\[
\begin{bmatrix}
0 & 0 & 0 \\
0 & 1 & 1 \\
0 & 1 & 1
\end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow x_2 + x_3 = 0 \Rightarrow x_2 = -x_3 \Rightarrow x = \begin{bmatrix} x_1 \\ -x_3 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}
\]

So there are two linearly independent eigenvectors:

\[
v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}
\]

We might as well take eigenvectors

\[
\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \text{ or } \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}
\]

etc. There should be two linearly independent vectors, which form a basis of the corresponding nullspace.

Both algebraic multiplicity and geometric multiplicity of this eigenvalue is two.

Find eigenvectors corresponding to \(\lambda = 2\):

\[
\begin{bmatrix}
-2 & 0 & 0 \\
0 & -1 & 1 \\
0 & 1 & -1
\end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow -2x_1 = 0, \ x_2 - x_3 = 0 \Rightarrow x_1 = 0, \ x_2 = x_3.
\]

So the only eigenvector is

\[
v_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}
\]

The algebraic and geometric multiplicities of this eigenvalue are one. So this matrix is diagonalizable, it is not defective.