Decompose the function \( f(x) = \chi_{(1/3,1)} \) into the Fourier series on \([-1,1]\). Write the answer in sigma notation.

**Solution.** The Fourier series has the form

\[
f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos \pi nx + b_n \sin \pi nx).
\]

Here,

\[
a_n := \int_{-1}^{1} f(x) \cos \pi nx \, dx = \int_{1/3}^{1} \cos \pi nx \, dx = \frac{1}{\pi n} \sin \pi nx \bigg|_{x=1/3}^{x=1} = -\frac{\sin(\pi n/3)}{\pi n}, \quad n \geq 1;
\]

\[
a_0 = \int_{-1}^{1} f(x) \, dx = \frac{2}{3},
\]

\[
b_n = \int_{-1}^{1} f(x) \sin \pi nx \, dx = \int_{1/3}^{1} \sin \pi nx \, dx = \frac{1}{\pi n} (-\cos \pi nx \bigg|_{x=1/3}^{x=1}) = \frac{\cos(\pi n/3) - \cos n}{\pi n} = \frac{\cos(\pi n/3) - (-1)^n}{\pi n}, \quad n \geq 1;
\]

Answer:

\[
f(x) = \frac{1}{3} + \sum_{n=1}^{\infty} \left( -\frac{\sin(\pi n/3)}{\pi n} \cos(\pi nx) + \frac{\cos(\pi n/3) - (-1)^n}{\pi n} \sin(\pi nx) \right)
\]