11. Surface Integrals. February 6, 2013

11.1 Area of Small Patch

Consider the surface $\Sigma$ given by parametric equations $\mathbf{r} = \mathbf{r}(u,v)$, $x = x(u,v)$, $y = y(u,v)$, $z = z(u,v)$, $(u,v) \in D$, where $D$ is some region in $\mathbb{R}^2$. Consider the small patch of this surface corresponding to a small rectangle $[u, u + \Delta u] \times [v, v + \Delta v]$. Let us find its area $\Delta S$. This patch is approximately the parallelogram with vertices $\mathbf{r}(u,v)$, $\mathbf{r}(u + \Delta u, v)$, $\mathbf{r}(u + \Delta, v + \Delta v)$, $\mathbf{r}(u, v + \Delta v)$ based on vectors

\[
\mathbf{a} = \mathbf{r}(u + \Delta u, v) - \mathbf{r}(u,v) = \langle x(u + \Delta u, v) - x(u,v), y(u + \Delta u, v), z(u + \Delta u, v) - z(u,v) \rangle
\]

and

\[
\mathbf{b} = \mathbf{r}(u, v + \Delta v) - \mathbf{r}(u,v) = \langle x(u,v + \Delta v) - x(u,v), y(u,v + \Delta v), z(u,v + \Delta v) - z(u,v) \rangle.
\]

Since $\Delta u$ and $\Delta v$ are small, these vectors are approximately equal to $\mathbf{r}_u(u,v)\Delta u$ and $\mathbf{r}_v(u,v)\Delta v$, where

\[
\mathbf{r}_u = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle, \quad \text{and} \quad \mathbf{r}_v = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right\rangle.
\]

Its area is approximately

\[
|\mathbf{a} \times \mathbf{b}| \approx |\mathbf{r}_u \times \mathbf{r}_v|\Delta u \Delta v.
\]

Therefore, the small element $dS$ of the area is approximately equal to

\[
dS = |\mathbf{r}_u \times \mathbf{r}_v|dudv = |\mathbf{r}_u \times \mathbf{r}_v|dA.
\]

11.2. Surface Integrals

Let us return to the general case. The area of $\Sigma$ is equal to

\[
\iint_{\Sigma} 1dS = \iint_{D} |\mathbf{r}_u \times \mathbf{r}_v|dA
\]

For every function $f(x,y,z)$ we have:

\[
\iint_{\Sigma} f dS = \iint_{D} f(x(u,v),y(u,v),z(u,v))|\mathbf{r}_u \times \mathbf{r}_v|dA
\]

11.3. Unit Sphere Example

For the unit sphere

\[
x = \cos \theta \sin \varphi, \quad y = \sin \theta \sin \varphi, \quad z = \cos \varphi,
\]

we have:

\[
\mathbf{r}_\theta = \langle - \sin \theta \sin \varphi, \cos \theta \sin \varphi, 0 \rangle, \quad \mathbf{r}_\varphi = \langle \cos \theta \cos \varphi, \sin \theta \cos \varphi, - \sin \varphi \rangle.
\]

Cross-multiply them:

\[
\mathbf{r}_\varphi \times \mathbf{r}_\theta = \langle \sin^2 \varphi \cos \theta, \sin^2 \varphi \sin \theta, \sin \varphi \cos \varphi \rangle.
\]
The magnitude of this vector is
\[
\sqrt{\sin^4 \varphi \cos^2 \theta + \sin^4 \varphi \sin^2 \theta + \sin^2 \varphi \cos^2 \varphi} = \sqrt{\sin^4 \varphi + \sin^2 \varphi \cos^2 \varphi} = \sqrt{\sin^2 \varphi} = |\sin \varphi| = \sin \varphi \geq 0.
\]
Therefore, \(dS = \sin \varphi d\varphi d\theta\). Let us find its area. We have: \(D = \{0 \leq \theta < 2\pi, 0 \leq \varphi \leq \pi\}\). Thus, the surface area is
\[
\int\int_D |r_\varphi \times r_\theta| \, dA = \int_0^{2\pi} \int_0^\pi \sin \varphi d\varphi d\theta = \int_0^{2\pi} d\theta \int_0^\pi \sin \varphi d\varphi = 2\pi \cdot 2 = 4\pi
\]
Now, let us calculate the surface integral of the function \(f = z\) over this sphere:
\[
\int\int_\Sigma f \, dS = \int\int_D z |r_\varphi \times r_\theta| \, dA = \frac{1}{2} \int_0^{2\pi} d\theta \int_0^\pi \sin 2\varphi d\varphi = 0
\]
However, this follows from symmetry with respect to the \(xy\)-plane.

11.4. When Surface is a Graph

Assume \(\Sigma\) is a surface \(z = g(x,y), \ (x,y) \in D\), where \(D\) is some region in 2D. Then it can be parametrized as \(x = u, \ y = v, \ z = g(u,v), \ (u,v) \in D\). Therefore, \(r(u,v) = ui + vj + g(u,v)k\), and we have:
\[
r_u = \mathbf{i} + g_x \mathbf{k}, \quad \text{and} \quad r_v = \mathbf{j} + g_y \mathbf{k}.
\]
Therefore, \(r_u \times r_v = -g_x \mathbf{i} - g_y \mathbf{j} + \mathbf{k}\), and \(|r_u \times r_v| = \sqrt{g_x^2 + g_y^2 + 1}\). Thus,
\[
\text{Area}(\Sigma) = \int\int_D \sqrt{g_x^2 + g_y^2 + 1} \, dA
\]
and for any function \(f : D \rightarrow \mathbb{R}\), we have:
\[
\int\int_\Sigma f \, dS = \int\int_D f(x,y,g(x,y)) \sqrt{g_x^2 + g_y^2 + 1} \, dA
\]
**Example.** The area of the piece \(\Sigma\) of the plane \(z = x + 2y\) given by \(0 \leq x \leq 1, \ 0 \leq y \leq 1\): \(g(x,y) = x + 2y\), and \(D = [0,1] \times [0,1]\), so \(g_x = 1, \ g_y = 2\), and
\[
\text{Area}(\Sigma) = \int\int_D \sqrt{1^2 + 2^2 + 1} \, dA = \int\int_D \sqrt{6} \, dA = \sqrt{6} \text{Area}(D) = \sqrt{6}
\]