
14.1. Definition

Consider a parametric curve \( C \) given by
\[
\mathbf{r} = \mathbf{r}(t) : x = x(t), \ y = y(t), \ z = z(t), \ a \leq t \leq b.
\]
Take a field
\[
\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}.
\]
Define the line integral of \( \mathbf{F} \) along \( C \):
\[
\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P \, dx + Q \, dy + R \, dz = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt
\]
Or, in expanded form,
\[
\int_a^b \left[ P(x(t), y(t), z(t))x'(t) + Q(x(t), y(t), z(t))y'(t) + R(x(t), y(t), z(t))z'(t) \right] \, dt
\]

14.2. Example

Let \( C \) be the unit circle on the \( xy \)-plane, traversed once counterclockwise, starting and finishing at the point \((1, 0, 0)\). Then it can be parametrized by \( x = \cos t, \ y = \sin t, \ z = 0, \ 0 \leq t \leq 2\pi \). We have: \( \mathbf{r}(t) = \langle \cos t, \sin t, 0 \rangle \), and \( \mathbf{r}'(t) = \langle -\sin t, \cos t, 0 \rangle \). For \( \mathbf{F} = \langle x, y, z \rangle \), we have: \( \mathbf{F}(\mathbf{r}(t)) = \langle \cos t, \sin t, 0 \rangle \). Therefore,
\[
\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = \cos t(-\sin t) + \sin t \cos t = 0, \quad \text{and} \quad \int_C \mathbf{F} \cdot d\mathbf{r} = 0
\]
For \( \mathbf{F} = \langle -y, x, 0 \rangle \), we have: \( \mathbf{F}(\mathbf{r}(t)) = \langle -\sin t, \cos t, 0 \rangle \). Therefore,
\[
\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = (-\sin t)^2 + \cos^2 t = 1, \quad \text{and} \quad \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} dt = 2\pi
\]

14.3. Interpretation as Work

Assume \( \mathbf{F} \) is a force field, and the particle is moving along the curve \( C \). Then \( \int_C \mathbf{F} \cdot d\mathbf{r} \) is the amount of work done by this force.

Indeed, assume a constant force \( \mathbf{G} \) acts upon a point which moved along a straight line with displacement vector \( \mathbf{s} = \overrightarrow{AB} \), where \( A \) is the initial and \( B \) is the terminal position of the particle. Then the amount of work is \( \mathbf{G} \cdot \mathbf{s} \). This is the physical meaning of the dot product.

In our case, the force field is not constant, and the particle does not move along a straight line. So split the curve into small pieces. On each of them, the force field is almost constant, and the particle moves along almost a straight line. The role of \( \mathbf{s} \) is played by \( d\mathbf{r} \), and \( \mathbf{F} \) plays the role of \( \mathbf{G} \). So the amount of work done on any small piece is approximately \( \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} \). Sum these up and get the line integral.
14.4. Fundamental Theorem of Calculus

Much like the single-variable integral is an inverse operation for the single-variable derivative, the line integral is an inverse operation for the gradient. Consider a parametric curve $C$ given by equations $\mathbf{r} = \mathbf{r}(t), \ x = x(t), \ y = y(t), \ z = z(t), \ a \leq t \leq b$. Take a gradient vector field $\mathbf{F} = \nabla f$. Then

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

Indeed, by the Chain Rule

$$\int_C \nabla f \cdot d\mathbf{r} = \int_a^b \nabla f \cdot \mathbf{r}'(t) dt = \int_a^b \left( \frac{\partial f}{\partial x}(x(t), y(t), z(t)) \frac{dx}{dt} + \frac{\partial f}{\partial y}(x(t), y(t), z(t)) \frac{dy}{dt} + \frac{\partial f}{\partial z}(x(t), y(t), z(t)) \frac{dz}{dt} \right) dt = \int_a^b \frac{df(x(t), y(t), z(t))}{dt} dt = \int_a^b \frac{df(\mathbf{r}(t))}{dt} dt = f(\mathbf{r}(b)) - f(\mathbf{r}(a)).$$

**Example.** The force field $\mathbf{F} = \langle x, y, z \rangle$, considered in the first example, is conservative: $\mathbf{F} = \nabla f$, where $f = (x^2 + y^2 + z^2)/2$. Therefore, $a = 0$, $b = 2\pi$, and $\mathbf{r}(a) = \mathbf{r}(b)$: the initial and the terminal positions of the curve coincide. That’s why

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)) = 0$$