15. Line Integrals of Vector Fields II. February 13, 2013

15.1. Calculation Example

Find \( \int_{C} F \cdot dr \), where \( F(x, y) = e^{x-1}i + xyj \), \( C : r(t) = t^2i + t^3j \), \( 0 \leq t \leq 1 \).

This is one of the practice problems. We have: \( x(t) = t^2 \), \( y(t) = t^3 \), and

\[
r'(t) = <2t, 3t^2>, \quad F(r(t)) = e^{t^2-1}i + t^2t^3j = <e^{t^2-1}, t^5>
\]

Therefore,

\[
r'(t) \cdot F(r(t)) = 2te^{t^2-1} + 3t^7.
\]

We have:

\[
\int_{C} F \cdot dr = \int_{0}^{1} \left[ 2te^{t^2-1} + 3t^7 \right] dt = \int_{0}^{1} 2te^{t^2-1}dt + \left. \frac{3t^8}{8} \right|_{t=1}^{t=0} = \int_{-1}^{0} e^u du + \frac{3}{8} = 1 - e^{-1} + \frac{3}{8}
\]

Here, we let

\[
u = t^2 - 1, \quad du = 2tdt, \quad 0 \leq t \leq 1 \implies -1 \leq u \leq 0.
\]

15.2. Independence of Path Example

Show that the integral

\[
\int_{C} 2xe^{-y}dx + (2y - x^2e^{-y})dy,
\]

where \( C \) is any path from \((1, 0)\) to \((2, 1)\), is independent of this path, and find it. This is another practice problem. Independence of path means that if \( C_1 \) and \( C_2 \) are two paths from \((1, 0)\) to \((2, 1)\), then the integrals along them of this vector field are equal. We need to find a function \( f(x, y) \) such that

\[
F(x, y) = 2xe^{-y}i + (2y - x^2e^{-y})j = \nabla f.
\]

First of all, let us show that \( F \) is a conservative vector field (=a gradient vector field). It is sufficient to show that \( \text{curl} F = 0 \), see Lecture 13. We have:

\[
\text{curl} F = \begin{vmatrix}
i & j & k \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
2xe^{-y} & 2y - x^2e^{-y} & 0
\end{vmatrix} = \left( (2y - x^2e^{-y})_x - (2xe^{-y})_y \right) k = \left[ -2xe^{-y} + 2xe^{-y} \right] k = 0.
\]

Now, we need to find this function \( f \). This is done as follows:

\[
f_x = 2xe^{-y}, \quad f_y = 2y - x^2e^{-y},
\]

from the first equation, integrate with respect to \( x \):

\[
f(x, y) = \int 2xe^{-y}dx + g(y) = x^2e^{-y} + g(y),
\]

and then take a derivative with respect to \( y \):

\[
-x^2e^{-y} + g'(y) = 2y - x^2e^{-y}, \quad g'(y) = 2y, \quad g(y) = y^2.
\]
Therefore, 

\[ f(x, y) = y^2 + x^2 e^{-y}. \]

By the Fundamental Theorem of Calculus for Line Integrals, we have:

\[
\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f(2, 1) - f(1, 0) = (1 + 4e^{-1}) - 1 = 4e^{-1}
\]

The result does not depend on the curve \( C \), as long as it starts from \((1, 0)\) and ends at \((2, 1)\).