
16.1. Motivation and Definition

Assume we have a planar region \( \Sigma \) of area \( S \), with a normal vector \( n \), and a constant vector field \( F \). Then the flux or flow of this field through \( \Sigma \) is defined as

\[
(F \cdot n)S.
\]

It has the meaning of the rate of flow through \( \Sigma \). Indeed, assume \( F = v \) is the velocity of fluid flowing through the surface \( \Sigma \). How much volume will flow through \( \Sigma \) during time \( \Delta t \)? This volume has base \( \Sigma \) with area \( S \) and height \( \cos \theta |F| = n \cdot v \), where \( \theta \) is the angle between \( v \) and \( n \). So this volume is equal to \( S(n \cdot v)\Delta t \). Therefore, the rate of this flow is \( S(n \cdot v) \).

Now, assume we have a parametric surface \( \Sigma \), which is not necessarily a plane. It is given by parametric equations \( r = r(u,v), \ (u,v) \in D \), where \( D \subseteq \mathbb{R}^2 \) is some region. Assume we have a normal vector \( n(x,y,z) \) with length \( |n| = 1 \) assigned at each point \((x,y,z)\in \Sigma \). Assume that \( n(x,y,z) \) continuously depends on the point \((x,y,z)\). Consider a vector field \( F(x,y,z) \). Let us define the flux of the field \( F \) through \( \Sigma \), or the surface integral of \( F \) over \( \Sigma \):

\[
\int \int _{\Sigma} F \cdot dS = \int \int _{D} (F \cdot n) \, dA
\]

We split the surface \( \Sigma \) into small patches and calculate the flux through each patch, assuming it is planar. The notation \( dS \) stands for \( n \, dA \), this is "a small vector element of area".

This definition depends on orientation, that is, the choice of \( n \). There are two options, \( n_1 \) and \( n_2 \): on one side or on the other side of the surface. We always have: \( n_1 = -n_2 \), so these orientations are opposite. If a surface is closed, then we take by default \( n \) pointing outward. This is called positive orientation. So we always take a surface together with its orientation. If you switch to the other orientation, the surface integral changes its sign.

16.2. Calculation Formula

If we have a parametric surface \( r = r(u,v) \) which is smooth, i.e. \( r_u \) and \( r_v \) are not parallel, then we can take

\[
 n = \frac{r_u \times r_v}{|r_u \times r_v|}
\]

Indeed, \( r_u \) and \( r_v \) are tangent vectors to the surface, so their cross product is orthogonal to it. We need to divide by its magnitude to ensure \( |n| = 1 \). The element of surface area is \( dS = |r_u \times r_v| \, dA \), and

\[
\int \int _{\Sigma} F \cdot dS = \int \int _{D} F(r(u,v)) \cdot \frac{r_u \times r_v}{|r_u \times r_v|} \, r_u \times r_v \, dA = \int \int _{D} F(r(u,v)) \cdot (r_u \times r_v) \, dA
\]

16.3. Examples

1. Let \( F = k \), and \( \Sigma \) be the graph \( z = (x^2 + y^2)/2, \quad 0 \leq x,y \leq 1 \). Choose upward orientation, so that \( n \) points up. We have:

\[
r(u,v) = ui + vj + \frac{1}{2}(u^2 + v^2)k, \quad r_u = i + uk, \quad r_v = j + vk,
\]
and \( \mathbf{r}_u \times \mathbf{r}_v = < -u, -v, 1 > \). This normal vector indeed points up, because its third component is greater than zero. Otherwise, we needed to put a minus sign near this vector. Therefore, \( \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v) = 0 \cdot (-u) + 0 \cdot (-v) + 1 \cdot 1 = 1 \), and

\[
\iint_S \mathbf{F} \cdot d\mathbf{S} = \int_0^1 \int_0^1 1 \, dudv = 1
\]

2. Let \( \mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \), and suppose \( \Sigma \) is the cylinder \( x^2 + y^2 = 1, \ 0 \leq z \leq 1 \), with outward orientation. This is a parametric surface:

\[
\mathbf{r}(u, v) = < \cos u, \sin u, v >, \ 0 \leq u < 2\pi, \ 0 \leq v \leq 1.
\]

So \( D = \{ 0 \leq u < 2\pi, \ 0 \leq v \leq 1 \} \). We have:

\[
\mathbf{r}_u = < -\sin u, \cos u, 0 >, \ \mathbf{r}_v = < 0, 0, 1 >, \ \mathbf{r}_u \times \mathbf{r}_v = < \cos u, \sin u, 0 >.
\]

This is indeed an outward orientation. (Just draw a picture.) Also, \( \mathbf{F}(\mathbf{r}(u, v)) = < \cos u, \sin u, v > \), hence \( \mathbf{F}(\mathbf{r}(u, v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v) = \cos^2 u + \sin^2 u = 1 \). Thus,

\[
\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D 1 \, dA = \text{Area}(D) = 2\pi
\]