
19.1. Formulation

Here, we have only two dimensions: $x$ and $y$. Consider a curve $C$ in $\mathbb{R}^2$. Assume it is closed: it starts and ends at the same point. Assume it is traversed counterclockwise. Let $D$ be the region inside this curve. Take a 2D vector field $\mathbf{F}(x,y) = P(x,y) \mathbf{i} + Q(x,y) \mathbf{j}$. Then we have:

$$\int_C P(x,y) \, dx + Q(x,y) \, dy = \iint_D \left[ \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right] \, dA$$

By default, any closed curve in $\mathbb{R}^2$ is traversed counterclockwise.

19.2. Example 1

Let $C$ be the unit circle with the usual parametrization: $x = \cos t$, $y = \sin t$, $0 \leq t \leq 2\pi$. Let $P = y$ and $Q = x$. Then we have:

$$\int_C P \, dx + Q \, dy = \int_C y \, dx + x \, dy = \int_0^{2\pi} \sin t \, d\cos t + \cos t \, d\sin t = \int_0^{2\pi} d(\sin t \, \cos t) = \sin t \cos t \bigg|_{t=0}^{t=2\pi} = 0.$$

However, this can be calculated using Green’s Theorem:

$$\iint_D \left[ \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right] \, dA = \iint_D (1 - 1) \, dA = 0.$$

Sometimes it is easier to calculate a line integral, and sometimes, a double integral. The decision to use or not to use Green’s Theorem should be made on case-by-case basis.

19.3. Proof

Let us prove that

$$\int_C P(x,y) \, dx = -\iint_D \frac{\partial P}{\partial y} \, dA.$$

The other part, for $Q$, can be proved analogously. Let us first calculate the line integral. Assume $D$ is a type I domain: a region between two graphs of $x$, $D = \{a \leq x \leq b, \, g_1(x) \leq y \leq g_2(x)\}$. Then the boundary $C$ consists of four pieces. Two of them are vertical, so $x = \text{const}$, $dx = 0$ and $\int P \, dx = 0$ for them. Consider the piece $C_1 : y = g_1(x)$, $a \leq x \leq b$, the graph of the function $g_1$. Then we have:

$$\int_{C_1} P(x,y) \, dx = \int_a^b P(x,g_1(x)) \, dx.$$

For the remaining piece, $C_2 : y = g_2(x)$, $a \leq x \leq b$, the graph of the function $g_2$, we have:

$$\int_{C_2} P(x,y) \, dx = -\int_a^b P(x,g_2(x)) \, dx.$$

The minus sign stands here because we traverse in in the wrong direction, from right to left. Let us collect these results:

$$\int_C P(x,y) \, dx = \int_a^b [P(x,g_1(x)) - P(x,g_2(x))] \, dx.$$
Apply the Fundamental Theorem of Calculus:

\[ \int_a^b [P(x, g_1(x)) - P(x, g_2(x))] \, dx = - \int_a^b \int_{g_1(x)}^{g_2(x)} \frac{\partial P}{\partial y} \, dy \, dx = - \int_D \frac{\partial P}{\partial y} \, dA. \]

The proof is complete.

19.4. Example 2

A particle starts at \((-2, 0)\), moves along the \(x\)-axis to \((2, 0)\), and then along the semicircle \(y = \sqrt{4 - x^2}\) back to the starting point. Find the work \(W\) done by \(\mathbf{F}(x, y) = x \mathbf{i} + (x^3 + 3xy^2) \mathbf{j}\).

Denote this path by \(C\). Let \(D\) be the region enclosed by \(C\): the upper half-disc of radius 2. Then we have: \(P = x\), \(Q = x^3 + 3xy^2\), and

\[ W = \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D (Q_x - P_y) \, dA = \iint_D (3x^2 + 3y^2) \, dA. \]

Convert it to polar coordinates: \(D = \{0 \leq \theta \leq \pi, \ 0 \leq r \leq 2\}\), and \(dA = rdrd\theta\), \(x^2 + y^2 = r^2\). Therefore, we have:

\[ W = \int_0^\pi \int_0^2 3r^2 \, rdr \, d\theta = 3 \int_0^\pi \left[ \frac{r^4}{4} \right]_r^2 \, d\theta = 3 \pi \frac{2^4}{4} = 12 \pi \]

19.5. Domains with Holes

Actually, Green’s Theorem is true for domains with holes. In this case, the boundary consists of two or more separate disconnected components. The integral along the whole boundary is equal to the sum of the integrals along each component. You traverse the outer boundary counterclockwise, and inner boundaries clockwise. The general rule: when you move along any part of the boundary, the domain should be to your left.

Example. Let \(C_1\) and \(C_2\) be the circles of radiuses 1 and 2 respectively, traversed counterclockwise. Let \(D\) be the ring between them. Let \(C = C_2 - C_1\) be the boundary of \(D\), the union of these two circles. The minus sign stands here because we need to traverse the inner boundary, \(C_1\), clockwise, i.e. in the wrong direction. Then let \(P = y^2\), and \(Q = 3xy\). By definition, we have

\[ \int_C P \, dx + Q \, dy = \int_{C_1} P \, dx + Q \, dy - \int_{C_2} P \, dx + Q \, dy. \]

Calculate this integral using Green’s Theorem:

\[ \int_C P \, dx + Q \, dy = \iint_D \left[ \frac{\partial (3xy)}{\partial x} - \frac{\partial y^2}{\partial y} \right] \, dA = \iint_D (3y - 2y) \, dA = \int_D y \, dA = 0, \]

by symmetry. Indeed, the ring \(D\) is symmetric with respect to the \(x\)-axis, so the integrals over the upper half-ring and the lower half-ring cancel out. Therefore, we can state that these two integrals are equal (although we do not know their value!)

\[ \int_{C_1} P \, dx + Q \, dy = \int_{C_2} P \, dx + Q \, dy. \]