
2.1. Polar Coordinates

Sometimes it is convenient to write a point \((x, y)\) on the plane \(\mathbb{R}^2\) in polar coordinates \((r, \theta)\):

\[ x = r \cos \theta, \quad y = r \sin \theta, \quad r \geq 0, \quad 0 \leq \theta < 2\pi. \]

One of the reasons is that sometimes the curves are easier to write in polar than in Cartesian coordinates. We can recover \(r\) and \(\theta\) from \(x\) and \(y\) in the following way:

\[ r = \sqrt{x^2 + y^2}, \quad \cos \theta = \frac{x}{r}, \quad \sin \theta = \frac{y}{r}. \]

**Example.**

1. If \(r = 2\) and \(\theta = \frac{2\pi}{3}\), then \(x = 2 \cos(\frac{2\pi}{3}) = -1\) and \(y = 2 \sin(\frac{2\pi}{3}) = \sqrt{3}\).

2. If \(x = -\sqrt{3}\) and \(y = -1\), then \(r = \sqrt{(-\sqrt{3})^2 + (-1)^2} = \sqrt{3 + 1} = 2, \) and

\[ \cos \theta = \frac{x}{r} = -\frac{\sqrt{3}}{2}, \quad \sin \theta = \frac{y}{r} = -\frac{1}{2} \Rightarrow \theta = \frac{7\pi}{6}. \]

The equation \(r = r_0\) corresponds to the circle with radius \(r_0\) centered at the origin. The equation \(\theta = \theta_0 = \text{const}\) corresponds to the ray from the origin which forms the angle \(\theta_0\) with the \(x\)-axis. The equation \(r = a \cos \theta\), where \(a > 0\) is a constant, can also be written in Cartesian coordinates:

\[ x^2 + y^2 = r^2 = ar \cos \theta = ax, \]  
\[ \text{so } x^2 - ax + y^2 = 0, \]  
\[ \text{and } (x - a/2)^2 + y^2 = (a/2)^2. \]

This is the circle with radius \(a/2\) centered at the point \((a/2, 0)\). Similarly, \(r = a \sin \theta\) is a circle; one can easily calculate its center and radius.

2.2. The Change-of-Variable Formula

Take a domain \(D \subseteq \mathbb{R}^2\) and a function \(f : D \to \mathbb{R}\). Rewrite it in polar coordinates: \((x, y) \in D \Leftrightarrow (r, \theta) \in G\). Then

\[
\int \int_{D} f \, dA = \int \int_{G} f(r \cos \theta, r \sin \theta) r \, dr \, d\theta
\]

So we just need to rewrite the domain \(D\) and the function \(f\) in polar coordinates; and, finally, write \(dA = r \, dr \, d\theta\).

The last formula means that the area of a small polar rectangle \([r, r + dr] \times [\theta, \theta + d\theta]\) is approximately \(r \, dr \, d\theta\), when \(dr\) and \(d\theta\) are small. Sometimes we say that a small piece of area is \(r \, dr \, d\theta\). Let us explain why: this polar rectangle is approximately an ordinary rectangle with sides \(dr\) and \(r \, d\theta\). The last side corresponds to the piece of smaller circle: it has angular size \(d\theta\) and radius \(r\), so its length is \(r \, d\theta\).

2.3. Example

Take \(D = \{x \leq 0, \ x^2 + y^2 \leq 1\}\), let \(f = x^2\). Then, in polar coordinates,

\[ G = \{0 \leq r \leq 1, \ \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}\}, \quad f = r^2 \cos^2 \theta, \quad dA = r \, dr \, d\theta. \]
Therefore,

\[ \int\int_D f \, dA = \int_{\pi/2}^{3\pi/2} \int_0^1 r^2 \cos^2 \theta \cdot r \, dr \, d\theta = \int_{\pi/2}^{3\pi/2} \cos^2 \theta \, d\theta \int_0^1 r^3 \, dr = \int_{\pi/2}^{3\pi/2} \frac{1 + \cos 2\theta}{2} \cdot \frac{r^4}{4} \bigg|_{r=0}^{r=1} = \left( \frac{\theta}{2} + \frac{\sin 2\theta}{4} \right) \bigg|_{\theta=\pi/2}^{\theta=3\pi/2} \cdot \frac{1}{4} = \frac{\pi}{2} \cdot \frac{1}{4} = \frac{\pi}{8} \]