3. Triple Integrals

3.1. Physical Meaning of Double Integrals

Consider a lamina occupying a region \( D \subseteq \mathbb{R}^2 \) with density function \( \rho(x,y) \). Then its total mass is

\[
\int\int_D \rho(x,y) dA.
\]

Indeed, assume for simplicity that \( D \) is a rectangle. We can split it into small subrectangles \( R_1, \ldots, R_n \), and choose a point \((x_i, y_i)\) in each subrectangle. On each subrectangle \( R_i \), the density \( \rho(x,y) \) is almost constant: \( \rho(x,y) \approx \rho(x_i, y_i) \). The mass of \( R_i \) is approximately its area times this almost constant density: \( \rho(x_i, y_i) \text{Area}(R_i) \). Thus, the total mass is approximately

\[
\sum_{i=1}^{n} \rho(x_i, y_i) \text{Area}(R_i) \approx \int\int_D \rho(x,y) dA.
\]

3.2. Triple Integrals over Boxes

They are defined and calculated in much the same way as double integrals. First, assume \( B \) is a box, i.e. a region of the type

\[
B = [a,b] \times [c,d] \times [p,q] = \{a \leq x \leq b, c \leq y \leq d, p \leq z \leq q\}.
\]

Assume a function \( f : B \to \mathbb{R} \) is defined on this box. Let us split it into small sub-boxes \( B_1, B_2, \ldots, B_n \). Pick a point \((x_1, y_1, z_1)\) in \( B_1 \), \((x_2, y_2, z_2)\) in \( B_2 \), \ldots in each of these small boxes. Then

\[
\int\int\int_B f dV = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i, y_i, z_i) \text{Vol}(B_i),
\]

where the limit is taken as the sizes of boxes tend to zero. We can calculate it as an interated integral:

\[
\int\int\int_B f dV = \int_p^q \int_c^d \int_a^b f(x, y, z) dxdydz.
\]

We can take any order of integration, say first \( y \), then \( z \), then \( x \). The result will be the same.

You cannot visualize it as the region under the graph of \( f \), this would require 4D space. However, you can visualize it as a mass of lamina occupying the region \( B \) with density \( \rho(x,y,z) \) at the point \((x,y,z)\). This is similar to a double integral as the mass of a lamina.

3.3. Example 1

Evaluate \( \int\int\int_B yz dV \), where \( B = [0,1] \times [0,1] \times [0,1] \). We have:

\[
\int\int\int_B yz dV = \int_0^1 \left[ \int_0^1 \left[ \int_0^1 yz d\zeta \right] dy \right] dx = \int_0^1 \left[ \int_0^1 \frac{yz^2}{2} \bigg|_{z=0}^{z=1} \right] dy \ dx = \int_0^1 \left[ \int_0^1 \frac{y}{2} dy \right] dx = \int_0^1 \left[ \int_0^1 \frac{y}{4} \bigg|_{y=0}^{y=1} \right] dx = \int_0^1 \frac{1}{4} dx = \frac{x}{4} \bigg|_{x=0}^{x=1} = \frac{1}{4}.
\]
3.4. Triple Integrals over General Regions

Consider a bounded region \( E \subseteq \mathbb{R}^3 \) and a function \( f : E \to \mathbb{R} \). Let us define the \textit{triple integral of f over E}. Since \( E \) is bounded, enclose it into a box \( B \). Extend \( f \) onto \( B \) by assigning it zero values (inside \( B \) and outside \( E \)). By definition,

\[
\int \int \int_E f \, \text{d}V = \int \int \int_B f \, \text{d}V.
\]

The method of calculation is the same as for double integrals. Assume

\[
E = \{(x, y) \in D, \ u_1(x, y) \leq z \leq u_2(x, y)\}.
\]

Then

\[
\int \int \int_E f \, \text{d}V = \int \int_D \left[ \int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z) \, \text{d}z \right] \, \text{d}A
\]

The same can be done with the roles of \( x, y, z \) interchanged.

3.5. Example 2

Find \( \int \int \int_E zdV \), where \( E = \{x^2 + y^2 \leq 1, \ 0 \leq z \leq \sqrt{x^2 + y^2}\} \). Here, \( D = \{x^2 + y^2 \leq 1\} \), and \( u_1(x,y) = 0, \ u_2(x,y) = \sqrt{x^2 + y^2} \). This is equal to

\[
\int \int_D \left[ \int_0^{\sqrt{x^2+y^2}} z \, \text{d}z \right] \, \text{d}A = \int \int_D \frac{z^2}{2} \bigg|_{z=0}^{z=\sqrt{x^2+y^2}} \, \text{d}A = \int \int_D \frac{x^2 + y^2}{2} \, \text{d}A.
\]

Here, use polar coordinates: \( D = \{0 \leq r \leq 1, \ 0 \leq \theta \leq 2\pi\} \), and \( dA = rdrd\theta \). Therefore, we have:

\[
\int_0^{2\pi} \int_0^1 \frac{r^2}{2} \, r \, d\theta = \frac{1}{2} \int_0^{2\pi} d\theta \int_0^1 r^3 \, dr = \frac{1}{2} \cdot 2\pi \cdot \left. \frac{r^4}{4} \right|_{r=0}^{r=1} = \frac{\pi}{4}
\]