Axioms of Probability

Problem 1. Pick a random number from 1 to 1000. Find the probabilities of the following events:
   (i) This number is not divisible by any of the numbers 2, 3, 5?
   (ii) This number is not divisible by 3 but is divisible either by 2 or by 5?

Problem 2. Consider three events: $A$, $B$, $C$. Draw the diagram for each of the following events:
   
   $(A \setminus B) \setminus C$, $A \setminus (B \setminus C)$, $(A \cap (C \setminus B)) \cup C$,
   
   $(B \setminus A) \cup (A \setminus C)$, $A \cup (B \cap C)$, $A^c \setminus B$.

Problem 3. Consider the event $A \triangle B = (A \setminus B) \cup (B \setminus A)$. It is called the symmetric difference of $A$ and $B$ and means: either $A$ or $B$, but not both. Express its probability as a combination of $P(A)$, $P(B)$ and $P(A \cap B)$.

Problem 4. There are two computers and a printer. Consider the following events: $A = \{\text{first computer works}\}$, $B = \{\text{second computer works}\}$, $C = \{\text{the printer works}\}$. The system is functioning if at least one of the computers is working and the printer is working. Express this event in terms of $A$, $B$ and $C$.

Problem 5. When is the equality $A \cap B = A \cup B$ possible?

Problem 6. Roll a die $n$ times.
   (i) Calculate the probability $p(n)$ that you get at least one six.
   (ii) What is the minimal $n$ such that $p(n) > 99\%$?

Problem 7. (Variation of the Birthday Problem.) Suppose you take $n$ people and for each of them, the probability that he was born at any given month is $1/12$.
   (i) Calculate $p(n)$, the probability that some of them have the same month.
   (ii) What is the minimal $n$ such that $p(n) > 50\%$?

Problem 8. (from an actuarial exam) The probability that there are $n$ insured losses throughout a year obey the rule $p_{n+1} = p_n/5$. What is the probability that there are two or more insured losses?

Problem 9. We seat 10 people at the table: five men and five women. What is the probability that men sit together or women sit together?

Problem 10. A judge chooses a jury of $n$ people out of a pool of $N$ people. The choice is uniform among all subsets of $n$ elements of the pool. Unhappy with an outcome, he dissolves this jury (returning the people into the pool) and chooses another jury of $m$ people, independently of the first jury. What is the probability that the first jury and the second jury have $k$ common people?
(i) Solve the problem in general form.
(ii) Find the decimal value for \( N = 1000, n = m = 2, k = 1. \)

**Problem 11.** Roll a fair die 10 times. Compute the probability that at least one number occurs:
(i) exactly 9 times;
(ii) exactly 5 times.

**Problem 12.** (i) Suppose you wish to put numbers 1...\( n \) into \( m \) different bags. Empty bags are allowed. Find the number of ways to do this.
(ii) Find the sum of all multinomial coefficients

\[ \sum \binom{n}{i_1, \ldots, i_m}, \]

where the sum is over all integers \( i_1, \ldots, i_m \geq 0 \) such that \( i_1 + \ldots + i_m = n \). Do this in two ways: using the formula from Problem 13 of HW 1 and using logic and part (i).