Homework 4, due July 23

Random Variables

Problem 1. Let $X$ be the random number on a die: from 1 to 6.
(i) What is the distribution of $X$?
(ii) Calculate $E(X)$.
(iii) Calculate $E(X^2)$.
(iv) Calculate $Var(X)$.
(v) Calculate $E(X^3)$.
(vi) Calculate $E(\sin(\pi X))$.
(vii) Calculate $E(\cos(\pi X))$.
(viii) Calculate $E(2X - 4)$.

Problem 2. Throw a die twice and let $X,Y$ be the results.
(i) What is the distribution of $X + Y$?
(ii) Calculate $E(X + Y)$.
(iii) Calculate $E(2X - 3Y)$.
(iv) Calculate $E(XY)$.
(v) Calculate $E(X^2Y^2)$.
(vi) Calculate $Var(X + Y)$.
(vii) Calculate $Var(XY)$.
(viii) Find $Cov(X, X + Y)$.
(ix) Find $Cov(X - Y, X + Y)$.
(x) Are $X$ and $Y$ independent?
(xi) Are $X$ and $X + Y$ independent?

Problem 3. Show that for every random variable $X$ and every real numbers $a,b$, we have: $E(aX + b) = aE(X) + b$ and $Var(aX + b) = a^2 Var(X)$.

Problem 4. Toss a fair coin 3 times. Let $X$ be the total number of heads.
(i) What is the probability space $\Omega$?
(ii) What is the distribution of $X$?

Problem 5. (A game from the Wild West) You bet one dollar and throw three dice. If at least one of them is six, then I return you your dollar and give you as many dollars as the quantity of dice which resulted in six. For example, if there are exactly two dice which gave us six, then you win two dollars. If you do not get any dice with six, then you lose your dollar. So you either lose 1\$ or win 1\$, 2\$ or 3\$. What is the expected value of the amount you win?

Problem 6. An urn contains 11 balls, 3 white, 3 red, and 5 blue balls. Take out two balls at random, without replacement. You win $1 for each red ball you select and lose a $1 for each white ball you select. Determine the distribution of $X$, the amount you win.

Problem 7. An urn contains 10 balls numbered 1,\ldots,10. Select 2 balls at random, without replacement. Let $X$ be the smallest number among the two selected balls.
(i) Determine the distribution of $X$.
(ii) Find $\mathbf{E}X$ and $\text{Var}X$.
(iii) Find $\mathbf{P}(X \leq 3)$.

**Problem 8.** If $\text{Var}X = 0$ and $\mathbf{E}X = 3$, then what can you say about the random variable $X$?

**Problem 9*.** An insurance company determines that $N$, the number of claims received in a week, is a random variable with

$$
\mathbf{P}(N = n) = \frac{1}{2^{n+1}}, \quad n = 0, 1, 2, \ldots
$$

The company also determines that the number of claims received in a given week is independent of the number of claims received in any other week. Determine the probability that exactly 7 claims will be received during a given two-week period.

**Problem 10*.** The number of injury claims per month is modeled by a random variable $N$ with

$$
\mathbf{P}(N = n) = \frac{1}{(n+1)(n+2)}, \quad n = 0, 1, 2, \ldots
$$

Determine the probability of at least one claim during a particular month, given that there have been at most four claims during that month.

**Problem 11.** Consider the following two variables: $X$, $Y$, with distribution

$$
\mathbf{P}(X = -1, Y = 1) = 0.2, \quad \mathbf{P}(X = 2, Y = 1) = 0.3, \\
\mathbf{P}(X = -1, Y = 3) = 0.1, \quad \mathbf{P}(X = 2, Y = 3) = 0.4.
$$

(i) Find the distribution of $X$.
(ii) Find the distribution of $Y$.
(iii) Calculate $\mathbf{E}X$.
(iv) Calculate $\mathbf{E}Y$.
(v) Calculate $\mathbf{E}(XY)$.
(vi) Calculate $\mathbf{E}X^2$.
(vii) Calculate $\mathbf{E}Y^2$.
(viii) Calculate $\text{Var}X$.
(ix) Calculate $\text{Var}Y$.
(x) Calculate $\mathbf{E}(X^2Y)$.
(xi) Calculate $\text{Cov}(X, Y)$.
(xii) Are $X$ and $Y$ independent?
Review Exercises

Problem 12. How do we define the binomial coefficient \( \binom{n}{k} \)? Why is the following formula true?
\[
\binom{n}{k} = \frac{n!}{k!(n-k)!}.
\]

Problem 13. Explain why
\[
\binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n} = 2^n.
\]

Problem 14. An item is defective (independently of other items) with probability 0.3. You have a method of testing whether the item is defective, but it does not always give you correct answer. If the tested item is defective, the method detects the defect with probability 0.9 (and says the item is OK with probability 0.1). If the tested item is good, then the method says it is defective with probability 0.2 (and gives the right answer with probability 0.8). A box contains 3 items. You have tested all of them and the tests detect no defects. What is the probability that none of the 3 items is defective?

Problem 15. Roll a die 8 times. Find the probability that you get three 1, two 2 and three 3.

Problems from old actuarial exams are marked by a star.