Suppose you have a biased coin, which falls H with probability $p$, $0 < p < 1$. You toss it $N$ times. Let $X$ be the number of H. What is the distribution of the random variable $X$? We call this the sequence of Bernoulli trials: each trial can result in success (H) or failure (T), with probabilities $p$ and $q = 1 - p$ respectively.

Let $X_1 = 1$ if the first toss resulted in H, $X_1 = 0$ if the first toss resulted in T. Same for $X_2, X_3, \ldots$. Then

$$X = X_1 + X_2 + \ldots + X_N$$

The random variables $X_1, \ldots, X_N$ are independent and identically distributed: $P(X_i = 1) = p$, $P(X_i = 0) = q$. What is the probability that $X = k$? Let us consider the extreme cases: all H or all T.

$$P(X = 0) = q^N, \quad P(X = N) = p^N.$$ 

Now, consider the case of general $k$. We can choose which $k$ tosses out of $N$ result in H, there are $\binom{N}{k}$ choices. Each choice has probability $p^k q^{N-k}$. So

$$P(X = k) = \binom{N}{k} p^k q^{N-k}, \quad k = 0, \ldots, N$$

We denote the distribution of $X$ by $\text{Bin}(N, p)$: binomial distribution with parameters $N$ and $p$, or sequence of $N$ Bernoulli trials with probability $p$ of success. We write: $X \sim \text{Bin}(N, p)$.

**Example.** If we toss a fair coin three times, the number of H is $X \sim \text{Bin}(3, 1/2)$. It has the following distribution:

$$P(X = 0) = P(TTT) = \frac{1}{8},$$

$$P(X = 1) = P(TTH, THT, HTT) = \frac{3}{8},$$

$$P(X = 2) = P(THH, HTH, HHT) = \frac{3}{8},$$

$$P(X = 3) = P(HHH) = \frac{1}{8}.$$ 

Alternatively, we can just plug into the formula $N = 3, p = 1/2$.

**Example.** Toss twice a biased coin, when the probability of Heads is $p = 2/3$. The number of Heads is $X \sim \text{Bin}(2, 2/3)$. Therefore,

$$P(X = 0) = P(TT) = \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9},$$

$$P(X = 1) = P(TH, HT) = P(TH) + P(HT) = \frac{1}{3} \cdot \frac{2}{3} + \frac{2}{3} \cdot \frac{1}{3} = \frac{4}{9},$$

$$P(X = 2) = P(HH) = \frac{2}{3} \cdot \frac{2}{3} = \frac{4}{9}.$$ 

**Example.** An insurance company has $N = 10$ clients, and each buys flood insurance. The probability of having a flood for each client is $p = 2\%$, and floods for different clients are independent. In case of flood, the company will pay 100,000$. What premium does it need to require from clients?
The number of floods is $X \sim \text{Bin}(10, 0.02)$. For example,

$$P(X = 0) = 0.98^{10} = 82\%,$$

$$P(X \leq 2) = P(X = 0) + P(X = 1) + P(X = 2) = 0.98^{10} + \binom{10}{1}(0.98^9)(0.02^1) + \binom{10}{2}(0.98^8)(0.02^2) = 99\%.$$  

With high probability, there will be no more than two floods. Let us count on this, that the company has to pay 200,000$. But there are 10 clients, so each should be charged 20,000$. Of course, this premium is too high; nobody is going to pay such exorbitant prices. This is because the company has too few clients, and the risk is not diversified enough.

**Expectation and Variance.** We have:

$$EX = EX_1 + \ldots + EX_N.$$  

And $EX_1 = 1 \cdot p + 0 \cdot q = p$. Same for other $X_i$. So

$$EX = p + \ldots + p = Np.$$  

Since $X_1, \ldots, X_N$ are independent, we have:

$$\text{Var} X = \text{Var} X_1 + \ldots + \text{Var} X_N.$$  

Let us calculate $\text{Var} X_i$. We have: $\text{Var} X_i = E X_i^2 - (EX_i)^2$. But $EX_i^2 = P(X_i = 1) \cdot 1 + P(X_i = 0) \cdot 0 = p \cdot 1 + q \cdot 0 = p$. So $\text{Var} X_i = p - p^2 = p(1 - p) = pq$. Therefore,

$$EX = Np \quad \text{Var} X = Npq$$