Lecture 9. Independent Events

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Events $A$ and $B$ are called independent if knowledge of whether $A$ happened or not does not influence the probability of $B$:

$$P(B \mid A) = P(B).$$

Since

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)},$$

we can rewrite this as

$$P(A \cap B) = P(A)P(B).$$

**Example 1.** Toss the coin twice. Let $A = \{\text{first toss H}\}$, $B = \{\text{second toss H}\}$, $C = \{\text{both tosses the same}\}$. Then $A$ and $B$ are independent. Indeed, the probability space (the space of all outcomes) is

$$\Omega = \{HH, HT, TH, TT\}.$$

And we have:

$$A = \{HH, HT\}, \quad B = \{TH, HH\}, \quad C = \{TT, HH\}.$$

So

$$P(A \cap B) = P\{HH\} = \frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2} = P(A) \cdot P(B).$$

It is obvious that these events are independent, because they result in different tosses of the coin. In some other cases, it is not obvious. For example, $A$ and $C$ are also independent. Indeed,

$$P(A \cap C) = P\{HH\} = \frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2} = P(A) \cdot P(C).$$

Similarly, $B$ and $C$ are independent.

We can define independence for three or more events.

Events $A$, $B$ and $C$ are called independent if

$$P(A \cap B) = P(A)P(B), \quad P(A \cap C) = P(A)P(C), \quad P(B \cap C) = P(B)P(C),$$

and, in addition,

$$P(A \cap B \cap C) = P(A)P(B)P(C).$$

This last condition is important, because it does not automatically follow from the first three conditions. For example, if $A$, $B$ and $C$ are the events from the example, then $A$ and $B$ are independent, $B$ and $C$ are independent, $A$ and $C$ are independent, so these events are pairwise independent. But $A \cap B \cap C = \{HH\}$, so

$$P(A \cap B \cap C) = \frac{1}{4} \neq \frac{1}{8} = P(A)P(B)P(C).$$

**Example 2.** Consider Problem 2 from the last quiz. A person likes tea with probability 50%. He likes coffee with probability 60%. He likes both tea and coffee with probability 30%. What is the probability that he likes neither tea nor coffee?
Solution. Let $A = \{\text{likes tea}\}$, $B = \{\text{likes coffee}\}$. Then $A \cap B = \{\text{likes both tea and coffee}\}$. So

$$P(A) = 50\%,$$ $$P(B) = 60\%,$$ $$P(A \cap B) = 30\%.$$ 

Therefore, by the inclusion-exclusion formula

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 80\%,$$

and

$$P((A \cup B)^c) = 1 - P(A \cup B) = 20\%.$$ 

This is the answer, because $(A \cup B)^c = \{\text{likes neither tea nor coffee}\}$. Answer: 20%.

Note that $P(A^c) = 50\%$ and $P(B^c) = 40\%$. So

$$P(A^c \cap B^c) = P((A \cup B)^c) = 20\% = 40\% \cdot 50\% = P(A^c)P(B^c).$$ 

These events (that a person does not like tea and that he does not like coffee) are independent. Some students used this to solve the problem. But it is just a coincidence. This independence does not follow automatically. On the contrary, you need to use inclusion-exclusion formula to establish it. For example, if we set 70% for the probability that he likes coffee, then the events would not be independent.

Example 3. Toss a coin 10 times. If you know that exactly 7 Heads are tossed, what is the probability that your rst toss is Heads?

Solution. Let $A = \{\text{seven H are tossed}\}$, $B = \{\text{first toss H}\}$. Then $A \cap B = \{\text{first H and six H out of the next nine}\}$. There are $\binom{10}{7}$ ways to choose slots for seven tosses out of ten (the tosses which give H). The probability of each configuration is $1/2^{10}$. So

$$P(A) = \binom{10}{7} \cdot \frac{1}{2^{10}}.$$ 

Also,

$$P(A \cap B) = \frac{1}{2} \cdot \binom{9}{6} \cdot \frac{1}{2^9} = \frac{1}{2^{10}} \binom{9}{6}.$$ 

Therefore,

$$P(B | A) = \frac{P(A \cap B)}{P(A)} = \frac{\binom{9}{6} \binom{10}{7}}{\binom{10}{7}} = \frac{9 \cdot 8 \cdot 7}{10 \cdot 9 \cdot 8} = \frac{9}{10}.$$ 

Why is this not surprising? Conditioned on 7 Heads, they are equally likely to occur on any given 7 tosses. If you choose 7 tosses out of 10 at random, the rst toss is included in your choice with probability 7/10.